Derivable Maps on Alternative Rings

Aplicações Derivação sobre Anéis Alternativos

Bruno L. M. Ferreira
Universidade Tecnológica Federal do Paraná - COECI-UTFPR, Guarapuava, PR
brunoferreira@utfpr.edu.br

Ruth Nascimento
Universidade Tecnológica Federal do Paraná - COECI-UTFPR, Guarapuava, PR
ruthnascimento@utfpr.edu.br

Abstract: D is an application of an alternative ring in $\mathscr{R}$ satisfying $\mathscr{R}$ $D(ab) = D(a)b + aD(b)$ to the whole; $b \in \mathscr{R}$. With some conditions on $\mathscr{R}$, we show that D additive.

Keywords: prime alternative ring; idempotent element; derivable map; additive map.

Resumo: Seja D uma aplicação de um anel alternativo $\mathscr{R}$ em $\mathscr{R}$ satisfazendo $D(ab) = D(a)b + aD(b)$ para todo $a, b \in \mathscr{R}$. Com algumas condições sobre $\mathscr{R}$, mostramos que D é aditiva.

Palavras - Chave: anel alternativo primo; aplicação aditiva; aplicação derivação; elemento idempotente.

1 Introduction

In this paper, $\mathscr{R}$ will be a ring not necessarily associative or commutative. For $x, y, z \in \mathscr{R}$ we denote the *associator* by $(x, y, z) = (xy)z - x(yz)$.

A ring $\mathscr{R}$ is called k*−torsion free* if $kx = 0$ implies $x = 0$, for any $x \in \mathscr{R}$, where $k \in \mathbb{Z}$, $k > 0$, *prime* if $IJ \neq 0$ for any two nonzero ideals $I, J \subseteq \mathscr{R}$ and *semiprime* if it contains no nonzero ideal whose square is zero.

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A ring $\mathscr{R}$ is said to be alternative if

$$(x,x,y)=0=(y,x,x), \text{ for all } x,y\in\mathscr{R},$$

and flexible if

$$(x,y,x)=0, \text{ for all } x,y\in\mathscr{R}.$$

One easily sees that any alternative ring is flexible.
Let $\mathscr{R}$ be an alternative ring (not necessarily with identity element) and let $a,b\in\mathscr{R}$. Recall that for a 3—torsion free alternative ring $\mathscr{R}$ the following are equivalent:

(i) $\mathscr{R}$ is prime;

(ii) $(a\mathscr{R})b\neq 0$ for any nonzero $a,b\in\mathscr{R}$;

(iii) $a(\mathscr{R}b)\neq 0$ for any nonzero $a,b\in\mathscr{R}$.

You can find the proof in [1]. A mapping $D:\mathscr{R}\to\mathscr{R}$ is derivable (multiplicative derivation) if $D(ab)=D(a)b+aD(b)$ for all $a,b\in\mathscr{R}$. Let us consider $\mathscr{R}$ an alternative ring and let us fix a nontrivial idempotent $e_1\in\mathscr{R}$, i.e, $e_1^2=e_1$; $e_1\neq 0$ and e_1 is not an unity element. Let $e_2:\mathscr{R}\to\mathscr{R}$ and $e_2':\mathscr{R}\to\mathscr{R}$ be given by $e_2a=a-e_1a$ and $e_2'a=a-ae_1$. We shall denote $e_2'a$ by ae_2. Note that $\mathscr{R}$ need not have an identity element. The operation $x(1-y)$ for $x,y\in\mathscr{R}$ is understood as $x-xy$. It is easy to see that $(e_ia)e_j=e_i(ae_j)$ for all $a\in\mathscr{R}$ and $i,j=1,2$. Then $\mathscr{R}$ has a Peirce decomposition $\mathscr{R}=\mathscr{R}_{11}\oplus\mathscr{R}_{12}\oplus\mathscr{R}_{21}\oplus\mathscr{R}_{22}$, where $\mathscr{R}_{ij}=e_i\mathscr{R}e_j$ $(i,j=1,2)$, satisfying the multiplicative relations:

(i) $\mathscr{R}_{ij}\mathscr{R}_{jl}\subseteq\mathscr{R}_{il}$ $(i,j,l=1,2)$;

(ii) $\mathscr{R}_{ij}\mathscr{R}_{ij}\subseteq\mathscr{R}_{ji}$ $(i,j=1,2; i\neq j)$;

(iii) $\mathscr{R}_{ij}\mathscr{R}_{kl}=0$, if $j\neq k$ and $(i,j)\neq(k,l)$, $(i,j,k,l=1,2)$;

(iv) $x_{ij}^2=0$ for all $x_{ij}\in\mathscr{R}_{ij}$ $(i,j=1,2;\ i\neq j)$.

The study of the relationship between the multiplicative and the additive structures of a ring has become an interesting and active topic in ring theory. The present paper takes up the special case of an alternative ring. We investigate the problem of when a derivable map must be an additive map for the class of alternative rings.

2 The main theorem

We shall prove the following result:

Theorem 2.1: *Let $\mathscr{R}$ be an alternative ring containing a non-trivial idempotent e_1 and $\mathscr{R} = \mathscr{R}_{11} \oplus \mathscr{R}_{12} \oplus \mathscr{R}_{21} \oplus \mathscr{R}_{22}$, the Peirce Decomposition of $\mathscr{R}$, relative to e_1, satisfying:*

(i) If $((e_i a)e_j)x_{jk} = 0$ for all $x_{jk} \in \mathscr{R}_{jk}$, then $((e_i a)e_j) = 0$;

(ii) If $x_{ij}((e_j a)e_i) = 0$ for all $x_{ij} \in \mathscr{R}_{ij}$, then $((e_j a)e_i) = 0$,

for $i, j, k \in \{1, 2\}$. If $D : \mathscr{R} \longrightarrow \mathscr{R}$ is a multiplicative derivation, then D is additive.

The proof of the theorem is organized as a series of lemmas.

We begin with the following lemma with a simple proof.

Lemma 2.1: $D(0) = 0$.

Proof. $D(0) = D(0.0) = D(0)0 + 0D(0) = 0$. ◊

Lemma 2.2: $D(a_{ii} + a_{jk}) = D(a_{ii}) + D(a_{jk})$, $j \neq k$.

Proof. If $i = 1, j = 1, k = 2$, for any $t_{11} \in \mathscr{R}_{11}$, we compute $D(a_{11} + a_{12})t_{11} + a_{11}D(t_{11}) + D(a_{12}t_{11}) - D(a_{12})t_{11} = D(a_{11} + a_{12})t_{11} + (a_{11} + a_{12})D(t_{11}) = D((a_{11} + a_{12})t_{11}) = D(a_{11}t_{11}) = D(a_{11})t_{11} + a_{11}D(t_{11})$. By condition (i) of Theorem 2.1, we have $[D(a_{11}+a_{12})-(D(a_{11})+D(a_{12}))]_{11} = 0$ and $[D(a_{11}+a_{12})-(D(a_{11})+D(a_{12}))]_{21} = 0$. Now, for any $t_{22} \in \mathscr{R}_{22}$, we compute $D(a_{11} + a_{12})t_{22} + a_{12}D(t_{22}) + D(a_{11}t_{22}) - D(a_{11})t_{22} = D(a_{11} + a_{12})t_{22} + (a_{11} + a_{12})D(t_{22}) = D((a_{11} + a_{12})t_{22}) = D(a_{12}t_{22}) = D(a_{12})t_{22} + a_{12}D(t_{22})$. By condition (i) of Theorem 2.1, we have $[D(a_{11} + a_{12}) - (D(a_{11}) + D(a_{12}))]_{12} = 0$ and $[D(a_{11} + a_{12}) - (D(a_{11}) + D(a_{12}))]_{22} = 0$. Thus, $D(a_{11} + a_{12}) = D(a_{11}) + D(a_{12})$.

The proof of the remaining cases is similar. ◊

Lemma 2.3: $D(a_{12}+a_{21})=D(a_{12})+D(a_{21})$.

Proof. For any $t_{11}\in\mathscr{R}_{11}$, we compute $D(a_{12}+a_{21})t_{11}-D(a_{12})t_{11}+D(a_{21}t_{11})-D(a_{21})t_{11}=D(a_{12}+a_{21})t_{11}+(a_{12}+a_{21})D(t_{11})=D((a_{12}+a_{21})t_{11})=D(a_{21}t_{11})$. By condition (i) of Theorem 2.1, we have $[D(a_{12}+a_{21})-(D(a_{12})+D(a_{21}))]_{11}=0$ and $[D(a_{12}+a_{21})-(D(a_{12})+D(a_{21}))]_{21}=0$. Similarly, replacing t_{11} by t_{22} we obtain $[D(a_{12}+a_{21})-(D(a_{12})+D(a_{21}))]_{12}=0$ and $[D(a_{12}+a_{21})-(D(a_{12})+D(a_{21}))]_{22}=0$. ◊

Lemma 2.4: *(i)* $D(a_{12}+b_{12}c_{22})=D(a_{12})+D(b_{12}c_{22})$,

(ii) $D(a_{21}+b_{22}c_{21})=D(a_{21})+D(b_{22}c_{21})$.

Proof. (i) We have $(e_1+b_{12})(a_{12}+c_{22})=a_{12}+b_{12}a_{12}+b_{12}c_{22}$. Then by Lemma 2.2 we have $D[(e_1+b_{12})(a_{12}+c_{22})]=D(e_1+b_{12})(a_{12}+c_{22})+(e_1+b_{12})D(a_{12}+c_{22})=(D(e_1)+D(b_{12}))(a_{12}+c_{22})+(e_1+b_{12})(D(a_{12})+D(c_{22}))=D(e_1a_{12})+D(e_1c_{22})+D(b_{12}a_{12})+D(b_{12}c_{22})$. By other hand, by Lemma 2.3, $D(a_{12}+b_{12}a_{12}+b_{12}c_{22})=D(b_{12}a_{12})+D(a_{12}+b_{12}c_{22})$. Thus, $D(a_{12}+b_{12}c_{22})=D(a_{12})+D(b_{12}c_{22})$.

(ii) It is analogous, by using the relation $(a_{21}+b_{22})(e_1+c_{21})=a_{21}+a_{21}c_{21}+b_{22}c_{21}$. ◊

Lemma 2.5: *(i)* $D(a_{12}+b_{12})=D(a_{12})+D(b_{12})$,

(ii) $D(a_{21}+b_{21})=D(a_{21})+D(b_{21})$.

Proof. (i) For any $t_{22}\in\mathscr{R}_{22}$, by Lemma 2.4 part (i), we have $D(a_{12})t_{22}+a_{12}D(t_{22})+D(b_{12})t_{22}+b_{12}D(t_{22})=D(a_{12}t_{22})+D(b_{12}t_{22})=D(a_{12}t_{22}+b_{12}t_{22})=D((a_{12}+b_{12})t_{22})=D(a_{12}+b_{12})t_{22}+(a_{12}+b_{12})D(t_{22})$. Thus, by (i) of the Theorem 2.1, $[D(a_{12}+b_{12})-(D(a_{12})+D(b_{12}))]_{12}=0$ and $[D(a_{12}+b_{12})-(D(a_{12})+D(b_{12}))]_{22}=0$. Now, for any $t_{11}\in\mathscr{R}_{11}$, we have $D(a_{12}+b_{12})t_{11}+(a_{12}+b_{12})D(t_{11})=D((a_{12}+b_{12})t_{11})=D(0)=0=D(0)+D(0)=D(a_{12}t_{11})+D(b_{12}t_{11})=D(a_{12})t_{11}+a_{12}D(t_{11})+D(b_{12})t_{11}+b_{12}D(t_{11})$. So, by (i) of Theorem 2.1, $[D(a_{12}+b_{12})-(D(a_{12})+D(b_{12}))]_{11}=0$ and $[D(a_{12}+b_{12})-(D(a_{12})+D(b_{12}))]_{21}=0$.

(ii) For any $t_{22}\in\mathscr{R}_{22}$ by Lemma 2.4 part (ii), we have $D(t_{22}(a_{21}+b_{21}))=D(t_{22}a_{21}+t_{22}b_{21})=D(t_{22}a_{21})+D(t_{22}b_{21})=D(t_{22})a_{21}+t_{22}D(a_{21})+D(t_{22})b_{21}+$

$t_{22}D(b_{21}) = t_{22}(D(a_{21}) + D(b_{21})) + D(t_{22})(a_{21} + b_{21})$. By other hand, $D(t_{22}(a_{21} + b_{21})) = D(t_{22})(a_{21} + b_{21}) + t_{22}D(a_{21} + b_{21})$. Thus, by condition (ii) of Theorem 2.1, $[D(a_{21}+b_{21})-(D(a_{21})+D(b_{21}))]_{21} = 0$ and $[D(a_{21}+b_{21})-(D(a_{21})+D(b_{21}))]_{22} = 0$. Now, for any $t_{11} \in \mathscr{R}_{11}$, $D(t_{11})a_{21} + t_{11}D(a_{21}) + D(t_{11})b_{21} + t_{11}D(b_{21}) = D(t_{11}a_{21}) + D(t_{11}b_{21}) = 0 = D(0) = D(t_{11}(a_{21} + b_{21})) = D(t_{11})(a_{21} + b_{21}) + t_{11}D(a_{21} + b_{21})$. By condition (ii) of Theorem 2.1, $[D(a_{21} + b_{21}) - (D(a_{21}) + D(b_{21}))]_{11} = 0$ and $[D(a_{21} + b_{21}) - (D(a_{21}) + D(b_{21}))]_{12} = 0$. ◊

Lemma 2.6: *(i)* $D(a_{11} + b_{11}) = D(a_{11}) + D(b_{11})$,

(ii) $D(a_{22} + b_{22}) = D(a_{22}) + D(b_{22})$.

Proof. (i) For any $t_{22} \in \mathscr{R}_{22}$, we have $D(a_{11} + b_{11})t_{22} + (a_{11} + b_{11})D(t_{22}) = D((a_{11} + b_{11})t_{22}) = 0 = D(0) = D(0) + D(0) = D(a_{11}t_{22}) + D(b_{11}t_{22}) = D(a_{11})t_{22} + a_{11}D(t_{22}) + D(b_11)t_{22} + b_{11}D(t_{22})$. So, by condition (i) of Theorem 2.1, $[D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{12} = 0$ and $[D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{22} = 0$. Now, for any $t_{12} \in \mathscr{R}_{12}$ by Lemma 2.5 part (i), we have $D(a_{11}+b_{11})t_{12}+(a_{11}+b_{11})D(t_{12}) = D((a_{11} + b_{11})t_{12}) = D(a_{11}t_{12}) + D(b_{11}t_{12}) = D(a_{11})t_{12} + a_{11}D(t_{12}) + D(b_{11})t_{12} + b_{11}D(t_{12})$. Thus, by (i) of the Theorem, we have $[D(a_{11}+b_{11})-(D(a_{11})+D(b_{11}))]_{11} = 0$ and $[D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{21} = 0$.

(ii) It is analogous. ◊

Lemma 2.7: $D(a_{11} + b_{12} + c_{21} + d_{22}) = D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22})$.

Proof. For any $t_{11} \in \mathscr{R}_{11}$ by Lemma 2.2, we have $D(a_{11} + b_{12} + c_{21} + d_{22})t_{11} + (a_{11} + b_{12} + c_{21} + d_{22})D(t_{11}) = D((a_{11} + b_{12} + c_{21} + d_{22})t_{11}) = D(a_{11}t_{11} + c_{21}t_{11}) = D(a_{11}t_{11}) + D(c_{21}t_{11}) = D(a_{11})t_{11} + a_{11}D(t_{11}) + D(c_{21})t_{11} + c_{21}D(t_{11}) + 0 + 0 = D(a_{11})t_{11} + a_{11}D(t_{11}) + D(c_{21})t_{11} + c_{21}D(t_{11}) + D(b_{12}t_{11}) + D(d_{22}t_{11}) = D(a_{11})t_{11} + a_{11}D(t_{11}) + D(c_{21})t_{11} + c_{21}D(t_{11}) + D(b_{12})t_{11} + b_{12}D(t_{11}) + D(d_{22})t_{11} + d_{22}D(t_{11})$. Thus, by condition (i) of the Theorem 2.1, we have, $[D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{11} = 0$ and $[D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{21} = 0$. Now, for any $t_{22} \in \mathscr{R}_{22}$ by Lemma 2.2, we have $D(a_{11} + b_{12} + c_{21} + d_{22})t_{22} + (a_{11} + b_{12} + c_{21} + d_{22})D(t_{22}) = D((a_{11} + b_{12} + c_{21} + d_{22})t_{22}) = D(b_{12}t_{22} + d_{22}t_{22}) = D(b_{12}t_{22}) + D(d_{22}t_{22}) = D(b_{12})t_{22} + b_{12}D(t_{22}) +$

$D(d_{22})t_{22} + d_{22}D(t_{22}) + 0 + 0 = D(b_{12})t_{22} + b_{12}D(t_{22}) + D(d_{22})t_{22} + d_{22}D(t_{22}) + D(a_{11}t_{22}) + D(c_{21}t_{22}) = D(b_{12})t_{22} + b_{12}D(t_{22}) + D(d_{22})t_{22} + d_{22}D(t_{22}) + D(a_{11})t_{22} + a_{11}D(t_{22}) + D(c_{21})t_{22} + c_{21}D(t_{22})$. So, by condition (i) of Theorem 2.1, we have, $[D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{12} = 0$ and $[D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{22} = 0$. ◊

Now we are ready to prove our main result.

Proof of the Theorem 2.1: For any $a, b \in \mathcal{R}$, we write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. Applying Lemmas 2.5, 2.6 and 2.7, we have

$$\begin{aligned} D(a+b) &= D((a_{11}+a_{12}+a_{21}+a_{22})+(b_{11}+b_{12}+b_{21}+b_{22})) \\ &= D(a_{11}+b_{11}+a_{12}+b_{12}+a_{21}+b_{21}+a_{22}+b_{22}) \\ &= D(a_{11}+b_{11})+D(a_{12}+b_{12})+D(a_{21}+b_{21})+D(a_{22}+b_{22}) \\ &= D(a_{11})+D(b_{11})+D(a_{12})+D(b_{12})+D(a_{21})+D(b_{21})+D(a_{22})+ \\ &\quad D(b_{22}) \\ &= D(a_{11}+a_{12}+a_{21}+a_{22})+D(b_{11}+b_{12}+b_{21}+b_{22}) = D(a)+D(b). \end{aligned}$$

Therefore, D is additive. ◊

Corollary 2.1: *Let $\mathcal{R}$ be a 3-torsion free prime alternative ring containing a nontrivial idempotent and a multiplicative derivation $D : \mathcal{R} \longrightarrow \mathcal{R}$, then D is additive.*

Definition 2.1: *Let $\mathcal{R}$ be an alternative ring and d be a map from $\mathcal{R}$ into itself. We call d a Jordan derivable map if $d(ab + ba) = d(a)b + bd(a) + d(b)a + bd(a)$ for all $a, b \in \mathcal{R}$.*

Corollary 2.2: *Let $\mathcal{R}$ be an alternative ring containing a nontrivial idempotent, satisfying (i), (ii) of the Theorem 2.1 and a multiplicative derivation, $D : \mathcal{R} \longrightarrow \mathcal{R}$, then D is a additive Jordan derivable map.*

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