Derivable Maps on Alternative Rings

Aplicações Derivação sobre Anéis Alternativos

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Abstract: $D$ is an application of an alternative ring in $\mathcal{R}$ satisfying $\mathcal{R} \ D(ab) = D(a)b + aD(b)$ to the whole; $b \in \mathcal{R}$. With some conditions on $\mathcal{R}$, we show that $D$ additive.

Keywords: prime alternative ring; idempotent element; derivable map; additive map.

Resumo: Seja $D$ uma aplicação de um anel alternativo $\mathcal{R}$ em $\mathcal{R}$ satisfazendo $\mathcal{R} \ D(ab) = D(a)b + aD(b)$ para todo $a, b \in \mathcal{R}$. Com algumas condições sobre $\mathcal{R}$, mostramos que $D$ é aditiva.

Palavras - Chave: anel alternativo primo; aplicação aditiva; aplicação derivação; elemento idempotente.

1 Introduction

In this paper, $\mathcal{R}$ will be a ring not necessarily associative or commutative. For $x, y, z \in \mathcal{R}$ we denote the associator by $(x, y, z) = (xy)z - x(yz)$.

A ring $\mathcal{R}$ is called $k$–torsion free if $kx = 0$ implies $x = 0$, for any $x \in \mathcal{R}$, where $k \in \mathbb{Z}$, $k > 0$, prime if $IJ \neq 0$ for any two nonzero ideals $I, J \subseteq \mathcal{R}$ and semiprime if it contains no nonzero ideal whose square is zero.
A ring $R$ is said to be alternative if

$$(x, x, y) = 0 = (y, x, x), \text{ for all } x, y \in R,$$

and flexible if

$$(x, y, x) = 0, \text{ for all } x, y \in R.$$  

One easily sees that any alternative ring is flexible.

Let $R$ be an alternative ring (not necessarily with identity element) and let $a, b \in R$.

Recall that for a $3$–torsion free alternative ring $R$ the following are equivalent:

(i) $R$ is prime;

(ii) $(aR)b \neq 0$ for any nonzero $a, b \in R$;

(iii) $a(Rb) \neq 0$ for any nonzero $a, b \in R$.

You can find the proof in [1]. A mapping $D : R \to R$ is derivable (multiplicative derivation) if $D(ab) = D(a)b + aD(b)$ for all $a, b \in R$. Let us consider $R$ an alternative ring and let us fix a nontrivial idempotent $e_1 \in R$, i.e, $e_1^2 = e_1; e_1 \neq 0$ and $e_1$ is not an unity element. Let $e_2 : R \to R$ and $e'_2 : R \to R$ be given by $e_2a = a - e_1a$ and $e'_2a = a - ae_1$. We shall denote $e'_2a$ by $ae_2$. Note that $R$ need not have an identity element. The operation $x(1 - y)$ for $x, y \in R$ is understood as $x - xy$. It is easy to see that $(e_1a)e_j = e_1(ae_j)$ for all $a \in R$ and $i, j = 1, 2$. Then $R$ has a Peirce decomposition $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$, where $R_{ij} = e_ie_j$ $(i, j = 1, 2)$, satisfying the multiplicative relations:

(i) $R_{ij}R_{il} \subseteq R_{i1}$ $(i, j, l = 1, 2)$;

(ii) $R_{ij}R_{ij} \subseteq R_{i1}$ $(i, j = 1, 2; i \neq j)$;

(iii) $R_{ij}R_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l)$, $(i, j, k, l = 1, 2)$;

(iv) $x^2_{ij} = 0$ for all $x_{ij} \in R_{ij}$ $(i, j = 1, 2; i \neq j)$.

The study of the relationship between the multiplicative and the additive structures of a ring has become an interesting and active topic in ring theory. The present
paper takes up the special case of an alternative ring. We investigate the problem of when a derivable map must be an additive map for the class of alternative rings.

2 The main theorem

We shall prove the following result:

Theorem 2.1: Let \( R \) be an alternative ring containing a non-trivial idempotent \( e \) and \( R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22} \), the Peirce Decomposition of \( R \), relative to \( e \), satisfying:

(i) If \( ((e, a)e_j)x_{jk} = 0 \) for all \( x_{jk} \in R_{jk} \), then \( ((e, a)e_i) = 0 \);

(ii) If \( x_{ij}((e, a)e_i) = 0 \) for all \( x_{ij} \in R_{ij} \), then \( ((e, a)e_i) = 0 \),

for \( i, j, k \in \{1, 2\} \). If \( D : R \rightarrow R \) is a multiplicative derivation, then \( D \) is additive.

The proof of the theorem is organized as a series of lemmas.

We begin with the following lemma with a simple proof.

Lemma 2.1: \( D(0) = 0 \).

**Proof.** \( D(0) = D(0, 0) = D(0)0 + 0D(0) = 0 \). ◊

Lemma 2.2: \( D(a_{i1} + a_{jk}) = D(a_{i1}) + D(a_{jk}), \ j \neq k \).

**Proof.** If \( i = 1, j = 1, k = 2 \), for any \( t_{11} \in R_{11} \), we compute \( D(a_{i1} + a_{12})t_{11} + a_{11}D(t_{11}) + D(a_{i1}t_{11}) - D(a_{i1}t_{11}) = D(a_{i1} + a_{12})t_{11} + (a_{i1} + a_{12})D(t_{11}) = D(a_{i1} + a_{12})t_{11} \). By condition (i) of Theorem 2.1, we have \( [D(a_{i1} + a_{12}) - (D(a_{i1}) + D(a_{12}))]_{11} = 0 \) and \( [D(a_{i1} + a_{12}) - (D(a_{i1}) + D(a_{12}))]_{21} = 0 \). Now, for any \( t_{22} \in R_{22} \), we compute \( D(a_{i1} + a_{12})t_{22} + a_{12}D(t_{22}) + D(a_{i1}t_{22}) - D(a_{i1})t_{22} = D(a_{i1} + a_{12})t_{22} + (a_{i1} + a_{12})D(t_{22}) = D((a_{i1} + a_{12})t_{22}) = D(a_{i1}t_{22}) + a_{12}D(t_{22}) \). By condition (i) of Theorem 2.1, we have \( [D(a_{i1} + a_{12}) - (D(a_{i1}) + D(a_{12}))]_{22} = 0 \) and \( [D(a_{i1} + a_{12}) - (D(a_{i1}) + D(a_{12}))]_{22} = 0 \). Thus, \( D(a_{i1} + a_{12}) = D(a_{i1}) + D(a_{12}) \).

The proof of the remaining cases is similar. ◊
Lemma 2.3: \( D(a_{12} + a_{21}) = D(a_{12}) + D(a_{21}). \)

Proof. For any \( t_{11} \in \mathcal{R}_{11}, \) we compute \( D(a_{12} + a_{21})t_{11} - D(a_{12})t_{11} + D(a_{21}t_{11}) - D(a_{21})t_{11} = D(a_{12} + a_{21})t_{11} + (a_{12} + a_{21})D(t_{11}) = D((a_{12} + a_{21})t_{11}) = D(a_{21}t_{11}). \) By condition (i) of Theorem 2.1, we have \([D(a_{12} + a_{21}) - (D(a_{12}) + D(a_{21}))]_{11} = 0\) and \([D(a_{12} + a_{21}) - (D(a_{12}) + D(a_{21}))]_{21} = 0.\) Similarly, replacing \( t_{11} \) by \( t_{22} \) we obtain \([D(a_{12} + a_{21}) - (D(a_{12}) + D(a_{21}))]_{12} = 0\) and \([D(a_{12} + a_{21}) - (D(a_{12}) + D(a_{21}))]_{22} = 0.\)

Lemma 2.4:

(i) \( D(a_{12} + b_{12}c_{22}) = D(a_{12}) + D(b_{12}c_{22}), \)

(ii) \( D(a_{21} + b_{22}c_{21}) = D(a_{21}) + D(b_{22}c_{21}). \)

Proof. (i) We have \((e_1 + b_{12})(a_{12} + c_{22}) = a_{12} + b_{12}a_{12} + b_{12}c_{22}.\) Then by Lemma 2.2 we have \(D[(e_1 + b_{12})(a_{12} + c_{22})] = D(e_1 + b_{12})(a_{12} + c_{22}) + (e_1 + b_{12})D(a_{12} + c_{22}) = (D(e_1) + D(b_{12}))(a_{12} + c_{22}) + (e_1 + b_{12})(D(a_{12}) + D(c_{22})) = D(e_1a_{12}) + D(e_1c_{22}) + D(b_{12}a_{12}) + D(b_{12}c_{22}).\) By other hand, by Lemma 2.3, \(D(a_{12} + b_{12}a_{12} + b_{12}c_{22}) = D(b_{12}a_{12}) + D(a_{12} + b_{12}c_{22}).\) Thus, \(D(a_{12} + b_{12}c_{22}) = D(a_{12}) + D(b_{12}c_{22}).\)

(ii) It is analogous, by using the relation \((a_{21} + b_{22})(e_1 + c_{21}) = a_{21} + a_{21}c_{21} + b_{22}c_{21}.\)

Lemma 2.5:

(i) \( D(a_{12} + b_{12}) = D(a_{12}) + D(b_{12}). \)

(ii) \( D(a_{21} + b_{21}) = D(a_{21}) + D(b_{21}). \)

Proof. (i) For any \( t_{22} \in \mathcal{R}_{22}, \) by Lemma 2.4 part (i), we have \(D(a_{12})t_{22} + a_{12}D(t_{22}) + D(b_{12}t_{22}) + b_{12}D(t_{22}) = D(a_{12}t_{22}) + D(b_{12}t_{22}) = D(a_{12} + b_{12})t_{22} = D(a_{12} + b_{12}t_{22} + (a_{12} + b_{12})D(t_{22})).\) Thus, by (i) of the Theorem 2.1, \([D(a_{12} + b_{12}) - (D(a_{12}) + D(b_{12}))]_{12} = 0\) and \([D(a_{12} + b_{12}) - (D(a_{12}) + D(b_{12}))]_{22} = 0.\)

Now, for any \( t_{11} \in \mathcal{R}_{11}, \) we have \(D(a_{12} + b_{12})t_{11} + (a_{12} + b_{12})D(t_{11}) = D((a_{12} + b_{12})t_{11} + (a_{12} + b_{12})D(t_{11})) = D(0) = 0 = D(0) + D(0) = D(a_{12}t_{11}) + D(b_{12}t_{11}) = D(a_{12}t_{11} + a_{12}D(t_{11}) + D(b_{22}t_{11} + b_{22}D(t_{11})).\) So, by (i) of Theorem 2.1, \([D(a_{12} + b_{12}) - (D(a_{12}) + D(b_{12}))]_{11} = 0\) and \([D(a_{12} + b_{12}) - (D(a_{12}) + D(b_{12}))]_{21} = 0.\)

(ii) For any \( t_{22} \in \mathcal{R}_{22}, \) by Lemma 2.4 part (ii), we have \(D(t_{22}(a_{21} + b_{21})) = D(t_{22}a_{21} + t_{22}b_{21}) = D(t_{22}a_{21}) + D(t_{22}b_{21}) = D(t_{22})a_{21} + t_{22}D(a_{21}) + D(t_{22})b_{21} + D(t_{22})b_{21}.$$
Lemma 2.6: 

(i) \( D(a_{11} + b_{11}) = D(a_{11}) + D(b_{11}) \),

(ii) \( D(a_{22} + b_{22}) = D(a_{22}) + D(b_{22}) \).

**Proof.** (i) For any \( t_{22} \in \mathcal{R}_{22} \), we have \( D(a_{11} + b_{11})t_{22} + (a_{11} + b_{11})D(t_{22}) = D((a_{11} + b_{11})t_{22}) = 0 = D(0) = D(0) + D(0) = D(a_{11}t_{22}) + D(b_{11}t_{22}) = D(a_{11})t_{22} + a_{11}D(t_{22}) + D(b_{11})t_{22} + b_{11}D(t_{22}) \). So, by condition (i) of Theorem 2.1, \([D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{22} = 0 \) and \([D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{12} = 0 \). Now, for any \( t_{12} \in \mathcal{R}_{12} \) by Lemma 2.5 part (i), we have \( D(a_{11} + b_{11})t_{12} + (a_{11} + b_{11})D(t_{12}) = D((a_{11} + b_{11})t_{12}) = D(a_{11}t_{12}) + D(b_{11}t_{12}) = D(a_{11})t_{12} + a_{11}D(t_{12}) + D(b_{11})t_{12} + b_{11}D(t_{12}) \). Thus, by (i) of the Theorem, we have \([D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{11} = 0 \) and \([D(a_{11} + b_{11}) - (D(a_{11}) + D(b_{11}))]_{22} = 0 \).

(ii) It is analogous.

Lemma 2.7: \( D(a_{11} + b_{12} + c_{21} + d_{22}) = D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}) \).

**Proof.** For any \( t_{11} \in \mathcal{R}_{11} \) by Lemma 2.2, we have \( D(a_{11} + b_{12} + c_{21} + d_{22})t_{11} + (a_{11} + b_{12} + c_{21} + d_{22})D(t_{11}) = D((a_{11} + b_{12} + c_{21} + d_{22})t_{11}) = D(a_{11}t_{11} + c_{21}t_{11}) = D(a_{11}t_{11}) + D(c_{21}t_{11}) = D(a_{11}t_{11}) + a_{11}D(t_{11}) + D(c_{21}t_{11}) + c_{21}D(t_{11}) + \cdots + 0 = D(a_{11})t_{11} + a_{11}D(t_{11}) + D(c_{21})t_{11} + c_{21}D(t_{11}) + D(b_{12})t_{11} + D(d_{22})t_{11} \). Thus, by condition (i) of the Theorem 2.1, we have \([D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{11} = 0 \) and \([D(a_{11} + b_{12} + c_{21} + d_{22}) - (D(a_{11}) + D(b_{12}) + D(c_{21}) + D(d_{22}))]_{22} = 0 \). Now, for any \( t_{22} \in \mathcal{R}_{22} \) by Lemma 2.2, we have \( D(a_{11} + b_{12} + c_{21} + d_{22})t_{22} + (a_{11} + b_{12} + c_{21} + d_{22})D(t_{22}) = D((a_{11} + b_{12} + c_{21} + d_{22})t_{22}) = D(b_{12}t_{22}) = D(b_{12})t_{22} + b_{12}D(t_{22}) + D(d_{22})t_{22} = D(b_{12}t_{22} + d_{22}t_{22}) = D(b_{12}t_{22}) + D(d_{22})t_{22} = D(b_{12})t_{22} + b_{12}D(t_{22}) + D(d_{22})t_{22} \).
Definition 2.1: Let \( \mathcal{R} \) be an alternative ring and \( d \) be a map from \( \mathcal{R} \) into itself. We call \( d \) a Jordan derivable map if
\[
d(ab + ba) = d(a)b + b d(a) + d(b)a + b d(a)
\]
for all \( a, b \in \mathcal{R} \).

Corollary 2.2: Let \( \mathcal{R} \) be an alternative ring containing a nontrivial idempotent, satisfying (i), (ii) of the Theorem 2.1 and a multiplicative derivation, \( D : \mathcal{R} \rightarrow \mathcal{R} \), then \( D \) is a additive Jordan derivable map.
Referências

