

Living in Critters' world

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Abstract: We study several aspects of Critters, an invertible computation universal cellular automaton introduced by Norman Margolus in the 1980's. We review and extend analysis of some of its interesting, complex characteristics.

Key words: Cellular automata, Margolus neighborhood, lattice gases, diffusion, tunneling experiment, computer simulations of physical phenomena.

Resumo: Nós estudamos diversos aspectos de Critters, um autômato celular universal de computação inversa introduzido por Norman Margolus nos anos 80. Nós revisamos e extendemos a análise de algumas de suas características interessantes e complexas.

Palavras-chave: Autômatos celulares, vizinhança de Margolus, gás em rede, difusão, simulação computacional de fenômenos físicos.

1 Introduction

Cellular automata have shown to be capable to simulate a wide variety of physical phenomena that include, among others, chemical reactions, pattern formation, billiard ball computers and biological models [1, 2, 3, 4, 5, 6, 7, 8]. Moreover, some authors speculate that cellular automata can actually model our universe [9, 10, 11]. All these ideas and results emerged due to the fact that cellular automata, although simple in their definition, allow for incredibly complicated dynamics.

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In this work we focus on one particular cellular automaton called Critters and we try to relate its dynamics to the laws of physics. We explore the world of Critters by means of a large amount of computer simulations and numerical experiments. We also present theoretical ideas and models to explain its dynamical properties.

The idea of studying Critters rule was, in its origin, related to the simulation of a kind of ‘tunneling’ experiment in which a *particle* hits a *barrier* and we expect other *particles* to come out some time later. The Critters rule produces such an interesting behavior that it was worthwhile to study it and see what happens. The first image we had in mind concerning a tunneling experiment was to set up a barrier of tokens in a lattice and shoot particles onto it. Once a particle hits the barrier the system follows a well determined path until something comes out but what? In which direction? And at which time? There is certainly a lot of *uncertainty* in this experiment since it could happen that a particle bounces without penetrating the barrier or it could pass through it without any change; one kind of particle could hit the barrier and a different kind emerge or one come in and two or more come out!

In order to study this phenomenon we start describing the kind of particles we can construct, their masses and velocities and the way they interact with each other. Also we study different kinds of barriers and we make experiments of collisions between particles and barriers. In the last part we set up experiments to capture the diffusion properties of this rule.

1.1 The rule

A cellular automaton consists of a uniform array of cells each of which can have definite discrete states that change over time according to a definite recipe. The states of all cells are updated synchronously at discrete time steps based upon the previous states of cells in their neighborhoods. The application of the *transition rule* over time give rise to the evolution of these discrete *worlds*.

Critters is a *partitioning* cellular automaton, that is, a kind of cellular automaton in which the lattice is partitioned in blocks that are updated independently. Each cell can have two different states ‘occupied’ or ‘empty’ which we represent with black (or one) and white (or zero), respectively. When a site is occupied we say there is a ‘token’ in it.

The local interaction of a cellular automaton is of the form $f : Q^n \rightarrow Q$, i.e., n neighbors to 1 cell. On the contrary, the local interaction of a partitioning cellular automaton is of the form $f : Q^n \rightarrow Q^n$, i.e., n cells to n cells, where n is the number of cells of each partition or block in which the lattice is subdivided. Since every block acts as an independent cell, no communication happens between them. To allow the spreading of information between blocks, the lattice is shifted back and forth in time. Partitioning cellular automata are also known as cellular automata with the ‘Margolus neighborhood’ after Norman Margolus who introduced this technique to make it easier to capture physical properties (e.g., conserved quantities, invertibility

and symmetries) [13, 14, 15].

Figure 1 shows the partitioning scheme and the rule for Critters. There are sixteen different possible block configurations, however, the rule is rotation-invariant and only one representative of each case is shown; the four possible rotations of a block turn into the same rotation of the corresponding result-block¹.

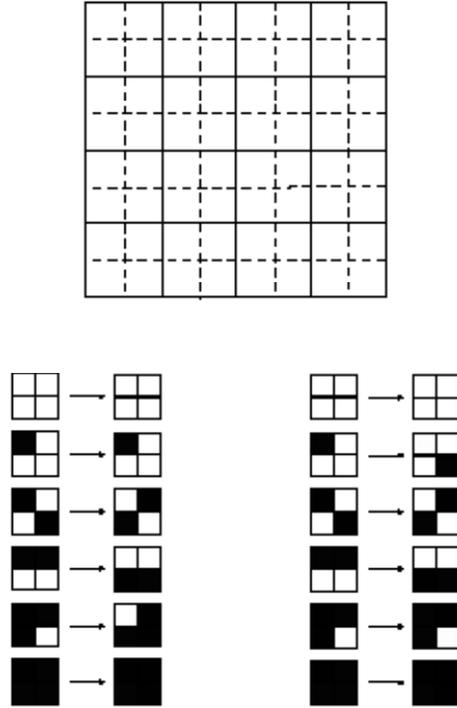
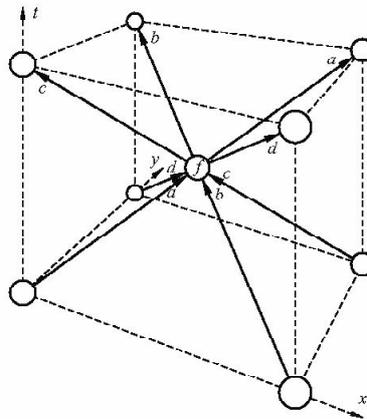


Figure 1 - Critters as a partitioning cellular automaton. In the upper part of the figure is an 8x8 lattice of a cellular automaton with the Margolus neighborhood; the dotted and solid sub-lattices are used at odd and even time steps, respectively. In the lower part is the transition rule for Critters; it depends on the time phase: on the left is the rule for odd time steps and on the right for even time steps.

When the lattice sites are treated in block in cellular automata, the picture is similar to lattice gases. Actually, there is a one to one correspondence between partitioning cellular automata and lattice gases. In lattice gases we have *signals* traveling in spacetime and places where groups of them interact at each time step. These places are processors and they form a mesh much like a *crystallographic* structure in spacetime. In the upper part of Figure 2 is the layout of a simple lattice gas. This structure is iterated over spacetime. In general the number of signals that enter a processor can be different from the number of signals coming out; in Critters

¹Note that, instead of complementing the data at each step as in the CAM book [14], here we conserve particles at each step, but achieve the same result by alternating between two variants of the rule on even and odd steps.

this number is 4 for both cases. The arcs are the tracks of the signals and can be ‘occupied’ or ‘empty’ represented with 1 and 0, respectively.



in	out	in	out
<i>abcd</i>	<i>abcd</i>	<i>abcd</i>	<i>abcd</i>
0000	0000	0000	0000
* 1000	0010	1000	1000
1100	1100	1100	1100
* 1010	0101	* 1010	0101
1110	1110	* 1110	1011
1111	1111	1111	1111

Figure 2 - Critters as a lattice gas. In the upper part, four signals a , b , c and d come in to a processor which applies the rule f and four signals come out. The evolution in time t is spanned in the vertical direction. In the lower part of the figure is the rule f expressed as a lookup table. Since by default the signals cross to the opposite corner at the next time step, we only need to specify the rules marked with an asterisk. Again, the left table is used in odd steps, the right one in even steps and because of rotational symmetry we show only one representative of each group of signal configurations. (Figure courtesy of Prof. Tommaso Toffoli, Boston University.)

The scheme shown in Figure 1 is the same as that of Figure 2 (seeing from above) in which the lattice sites represent the tracks of the signals. The dashed and solid sublattices show two sets of tracks for consecutive time steps and the processors are in the center of each block. From now on we will speak about tokens moving in a lattice or signals traveling in spacetime indistinctly.

If the local rule of a partitioning cellular automaton is invertible then, the global evolution of the system is automatically invertible, too. It is easy to check invertibility in Critters since it is a permutation, i.e., the rule maps the set of all block configurations to itself and each one of them to a different successor. If we simply apply the same part of the rule (odd or even steps) twice and then continue alternating the updates, the dynamics is reversed. Note that the rule is not only reversible but also symmetric with respect to zeros and ones, that is, if at a given time step we replace all zeros by ones and all ones by zeros (and apply two times the same part of the rule), the system will continue running in *negative*.

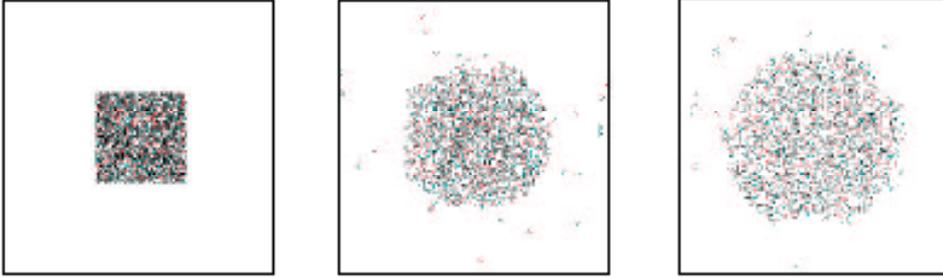


Figure 3 - The width of the lattice is 256 cells and periodic boundary conditions are set in all directions so the topology of the space is that of a torus. On the left is the configuration at time 0 and in the center a snapshot after 1000 time steps of evolution. The small particles are ‘gliders’ that move in straight lines horizontally or vertically. On the right is the state of the system after 5000 time steps. Note the increase in the diameter of the blob.

Figure 3 shows the evolution of the system for a particular initial configuration. We set up an empty region in which a square is randomly filled with tokens. After some time the square is deformed to a circular region. At the same time ‘gliders’ come out of the blob and start bouncing each other and producing oscillating structures, turning or disappearing inside the blob. The evolution in real time resembles a hive in which bees are coming and going and walking over each other. It can also be seen that the blob spreads out as time passes by.

2 Microscopic dynamics

If we put a few tokens in a lattice like that of Figure 1 and apply the rules we can follow their movement in detail. In this part we describe some of the microscopic structures, oscillators, particles and barriers we can construct. We give the relationship between *masses* (number of tokens), period and velocity of infinite sets of particles. We also study collisions between particles and with barriers and give the frequency distribution of the emission time for a particular case.

2.1 Oscillators

As in the well-known game Life [16] we find time-invariant structures that oscillate with a certain period. The simplest of these structures is a solitary token. As can be seen in the rule of Figure 1 a solitary token remains in place at odd time steps and moves diagonally at even time steps going back and forth between two diagonal positions and returning to its initial state after four time steps. If we put two tokens side by side, as in Figure 4a, they form an eight cycle but if we shift them one place to the right or to the left they become a four cycle². If we put three tokens in a row they form an eight cycle independently of the position with respect to the grid. In general, a *segment* forms oscillators that can have only determined periods depending on the number of tokens that compose them.

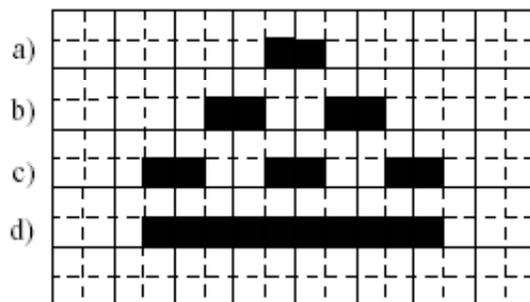


Figure 4 - Oscillatos: (a) two tokens side by side with period eight; (b) and (c) are dashed oscillators of period 16 and 24 respectively; in (d) the oscillator is made of the two components in (b) and (c) and its period is given by the $l.c.m.(16, 24) = 48$.

If we place an *odd* number i of tokens side by side they form oscillators with period $i^2 - 1$. On the other hand, if we attach an *even* number of tokens j we have two different cases:

a) If the number of tokens is a multiple of 4, that is, $j = 4, 8, 12, 16, \dots$ the line of tokens can have two different modes of oscillation depending on the position in the lattice. The edges of the line can be entirely inside a block or straddling on two blocks but both modes have period $2j$.

b) If the number of tokens is of the form $j = 2, 6, 10, 14, 18, \dots$ the structure can have also two modes of oscillation but now with different periods. One of the modes has period $2j$ and the other is formed by *coupling* of two different oscillators. For example, if the structure of Figure 4d is shifted one site to the right (or to the left) its period becomes $2j = 20$ steps but when all the tokens fit in the blocks as shown in the figure we have a different behavior.

²Unless otherwise stated all the movements start in an *odd* step, i.e., involving the dashed sub-lattice and the rules on the left in Figure 1.

In order to understand this case we have to examine first a different kind of oscillator made up of a *dashed* line of tokens. They are composed of segments of 2 tokens separated by 2 empty cells as in Figure 4b and Figure 4c. These are oscillators with a period that is four times the number of tokens. We can continue making longer lines with the same repetitive structure obtaining larger oscillators with correspondingly larger periods. They all behave like a string with fixed end points that has been plucked at the middle. After a few steps the string reaches its maximum extension (forms a triangular shape) and it bounces back to its initial position now going in the opposite direction and so on.

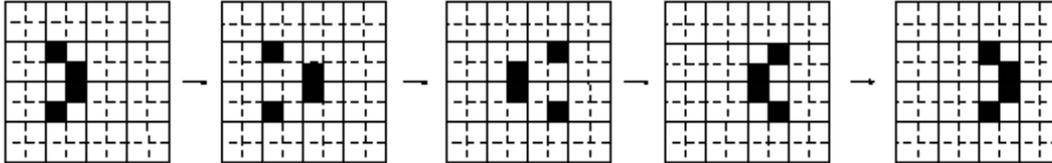


Figure 5 - The most common type of glider. From left to right we see that after four time steps the structure comes back to its initial configuration, i.e., its period is 4. On each cycle it travels two units along the lattice. Its velocity is $1/2$ the maximum attainable, i.e., half the velocity of light.

Now we can explain the second mode of case b) above. It can be seen that the solid line of Figure 4d is composed of the two structures of Figure 4b and Figure 4c and its period is exactly the *least common multiple* of the period of the two components, that is, 48. For example, when the solid line is 34 tokens wide and the edges fit completely inside a block, the structure is composed of two strings fixed at the edges, one with 18 tokens and period 72 and the other composed of 16 tokens with period 64. Therefore, the period of the line is $l.c.m.(72, 64) = 576$ time steps. This relationship is a general result and the periods can also be obtained with the following simple formula: $(1/2)j^2 - 2$ where j is the number of tokens in the line.

The world of Critters has a rich structure and we have just started to unravel a small part of it. There are infinitely many of these structures of all kinds and number of tokens. We will see more examples of large oscillators when talking about stable barriers in 2.4.

2.2 Particles

There are other structures known as ‘gliders’ that *move* around the lattice; they are also invariant under a ‘spacetime translation’. These particles have a certain velocity given by the ratio between the number of lattice sites traveled during a cycle and the time period.

Figure 5 shows the typical glider. It is made up of four tokens and walks two lattice sites every four time steps. If we look at two consecutive cycles it is easy to

see that this glider is composed of two oscillators of period 8 like the one of Figure 4a. These two oscillators act independently of each other during three of the 4 steps shown in the figure; only in the 4th stage they work together to make the structure come back to the original configuration. The movement of this glider resembles that of a frog squatting and extending its arms and legs: at the first application of the rule the two tokens on the front (arms) are in a separate block and cross to the other side while the two tokens in the back (legs) are isolated and remain in their places. On the second step the four tokens are in separate solid blocks and the rule mandate to cross to the opposite corner so, the two in the back come together to the center and the two in the front separate. On the third step we apply the odd rule again, the legs are in a dashed block and move forward while the arms in the front stay in place. Finally, we are in an even step and have two blocks of the same kind with the opposite corners occupied, the rule indicates ‘to flip positions’ and we are back to the first configuration.

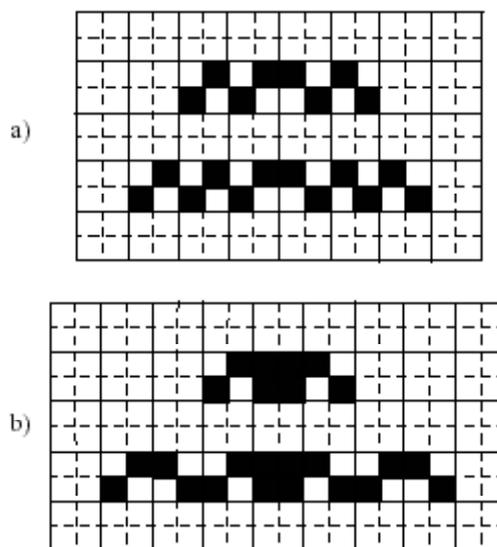


Figure 6 - Examples of gliders all moving upwards: (a) In the upper part we have the glider-seed with ‘single ballast’ that makes it slower. It has period 16 and velocity $1/8$ th of the velocity of light. In the lower part is the glider-seed with ‘double ballast’. In this case the period is 28 and it moves at $1/14$ th of the velocity of light. (b) In the upper part is another kind of glider that travels at half the speed of light and in the lower part we added two glider-seed at its sides reducing the velocity to $1/4$ th of c .

The glider of Figure 5 can be used as a *seed* or *unit* to construct other kinds of gliders with different velocities. We can put the units in tandem forming *trains* or *worms* that also travel at half the speed of light no matter how many *carts* are added. Two slightly different and slower gliders can be seen in Figure 6a. They are constructed adding to the glider-seed four and eight particles at its sides, respectively. If we try to add more ‘ballast’ in this way the structure becomes unstable,

shoot some glider and part of it becomes an oscillator.

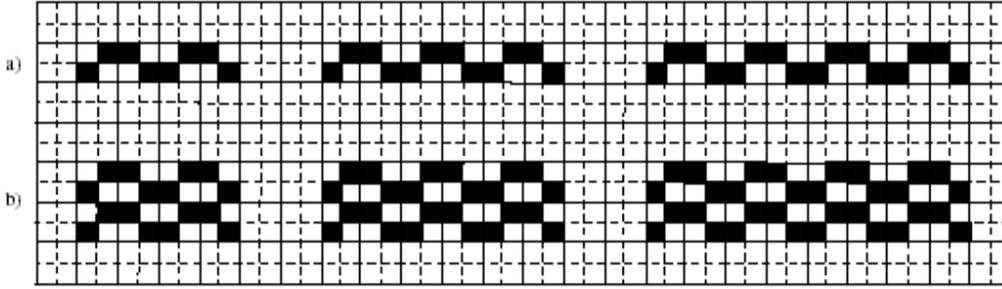


Figure 7 - (a) Three gliders obtained by adding the units of Figure 5 side by side. On the left we have a glider made of two units; it has period 8 and velocity $1/4$ th of c . In the center, a glider of period 12 and velocity equal to $1/6$ th of c . The one on the right, made of four units, has a period of 16 time steps and travels at $1/8$ th of c . The relationship between the number of tokens, the period and velocity is given in Table 1. (b) Adding two rows we can get a different set of gliders. Their velocities are the same as in (a) but their masses have been doubled. The process can be extended in the horizontal direction forever, however, if we add more rows the structures become unstable.

units	standard form			with 'single ballast'		
	m	θ	v	m	θ	v
1	4	4	$1/2$	8	16	$1/8$
2	8	8	$1/4$	12	28	$1/14$
3	12	12	$1/6$	16	40	$1/20$
4	16	16	$1/8$	20	52	$1/26$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
k	$4k$	$4k$	$1/2k$	$4(k+1)$	$12k+4$	$1/(6k+2)$

Table 1 - Relationship between the number of tokens (or mass) m , the period θ and the velocity v (given as a fraction of c) of the gliders of Figure 7a in 'standart form' and with 'single ballast' as in the upper part of Figure 6a.

In order to extend this kind of particle to any size and at the same time have different velocities we have to join them side by side as shown in Figure 7a. When we add units in this way we obtain slower particles. For example, if we pretend that the glider-seed actually travels at half the speed of light ($\approx 150000 : km/s$) then, in order to construct a glider that travels at the velocity of a commercial plane ($\approx 1000 : km/h$) we need to add more than half a million units! We can also add 'single ballast' to any of the gliders in Figure 7a to make new and slower particles. It should be noted that these are also built by coupling *dashed* oscillators as that of Figure 4. Table 1 shows the characteristics of this family of gliders.

In some cases, for example, when we have two or three glider-seeds side by side and we add ‘double ballast’ (lower part of Figure 6a) the structure agitates for a moment and becomes unstable, it shoots one or more particles and turns into an oscillator. These larger weights can be added only when the number of units attached side by side is of the form $3i + 1$ with $i = 0, 1, 2, \dots$. Table 2 shows their characteristics. If we recalculate the number of units for a glider that travels at $\approx 1000 : km/h$ now using ‘double ballast’ we *only* need about 100000 units, i.e., approximately five times fewer.

units	with ‘double ballast’		
	m	θ	v
1	12	28	1/14
4	24	88	1/44
7	36	148	1/74
10	48	208	1/104
\vdots	\vdots	\vdots	\vdots
k	4k+8	20k+8	1/(10k+4)

Table2 - Relationship between the mass m , the period θ and the velocity v of the gliders of Figure 7a with ‘double ballast’ as in the lower part of Figure 6a.

Figure 7b shows another generalization scheme. Joining two rows the particles conserve the same period and velocity as the ones in Figure 7a, however, adding three or more rows make them unstable. We have tried adding three and four rows in a number of cases and the structures oscillate during a few steps and start shooting gliders of different kinds and in different directions remaining as oscillators.

The gliders of Figure 6b are another possible set of particles that can be made as long as we want adding pairs of glider-unit to their sides. They all travel two cells on a cycle and their velocities are given by $v = 1/2k$ in which k now represents the number of added pairs.

Other combinations and different structures are yet to be found; the actual space of possibilities seems infinite and we already have enough material to turn our attention to a different aspect of this world. We can ask, for example, what happens when two of these particles collide? Do they follow the laws of physics?

2.3 Physical and non-physical collisions

One of the most general laws of physics is the conservation of momentum of a closed system. This means that, if we take into account all the particles involved in a collision, we should get the same amount of momentum before and after the impact. However, in the world of Critters this is not always the case. In Figure 8 we show four possible collisions between unit gliders. The first three cases can be

considered common examples we can find in our world between balls of billiard but in Figure 8d we have a completely different situation.

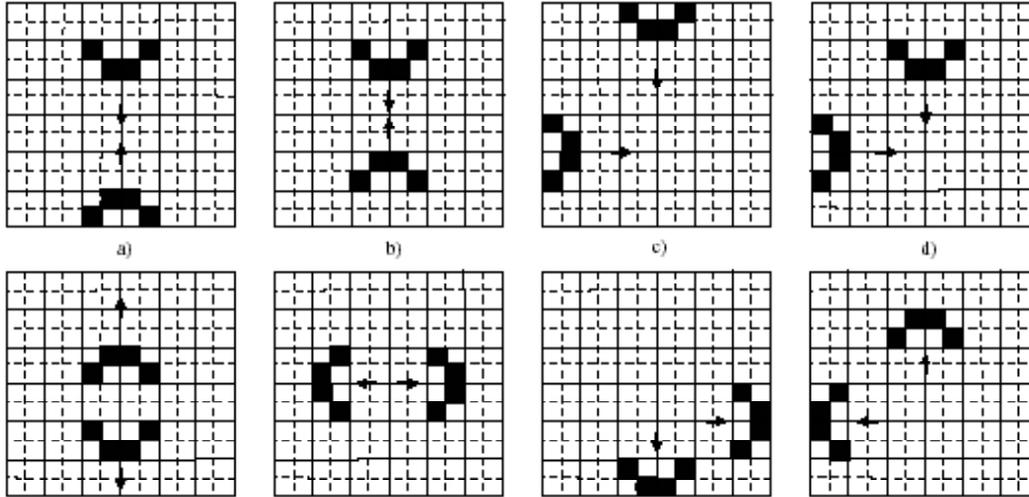


Figure 8 - The upper and lower rows of figures display the configuration of the system before and after the collision, respectively. The arrows indicate the direction of movement. (a) A head-on collision in which the gliders bounce back to where they come from. In the lower part is the configuration after 12 time steps. In (b) a head-on collision in which the gliders are scattered perpendicularly. The configuration after 6 time steps is shown in the lower row. In (c) they approach each other and come out of the impact at 90° . The configuration in the lower row is after 14 time steps. In (d) momentum is *reversed* and the particles come back on their tracks! In the lower row is shown the configuration after 14 steps.

Imagine that two cars are approaching an intersection at high speed, one from the North and the other from the West say, downtown Boston. Now, suppose both drivers think ‘today I feel lucky, the intersection will be free so I’m not gonna slow down’. You are watching the scene from a nearby cafeteria enjoying the day; suddenly, you see the cars hitting each other at a right angle, make a 180° turn and come out of the crash in the direction in which they were coming! Strange accidents can certainly happen in the Boston traffic, specially during rush hours, but this crash can never be seen since it does not conserve momentum. Well, that is what happens in the collision of Figure 8d.

We have done other experiments with particles of different velocities and obtained many different outcomes. In one of them, for example, four glider-units approach each other, one from the North, one from the South, one from the East and the other from the West. The result is one glider made of two units traveling South and a glider-seed with single ballast traveling North. The exact position of the gliders with respect to the lattice, the relative position between them and the

time phase define different cases.

Finally, it should be stressed that since the rule is invertible, all the collisions can be seen in reverse, i.e., if we change the direction of the arrows and watch the lower rows of pictures as the initial configuration then the result of the collisions is in the upper row in Figure 8.

2.4 Experiments with barriers

As we mention in the introduction, if we shoot a glider into a barrier it would be very difficult to predict the exact outcome of the experiment. However, we can first explore the possibilities in a *theoretical* manner since there are some general questions already answered before getting started.

In a finite space, any cellular automaton must eventually enter a cycle because it will run out of new states. If the rule is also reversible, the system will come back to its original state because each state can only have one predecessor [17].

Suppose we have a finite two dimensional lattice with periodic boundary conditions. Since the space is finite and Critters is invertible, the system is on a cycle. We put a stable barrier, i.e., a structure that wraps around the torus in the horizontal direction x and remains bounded in the vertical direction y forever. In other words, the *clump* is an oscillator and parts of it can move back and forth in the vertical direction but it will never shoot a particle. Then we place a glider heading towards the barrier outside the limits of the barrier and let the program run. The barrier and the glider form a new system with a different period. The glider will hit the barrier and after some finite time we will see a glider of the same kind coming out from the other side and going all the way around the torus to return to the place where we initially put it. However, the cycle could be large enough to allow other interesting things to happen in between. For example, other gliders can come out and hit each other creating oscillators and showing a rich evolution.

Now suppose the lattice is infinite in the y direction and finite in the x direction. We have a stable horizontal barrier with a certain finite period and shoot a glider at it. We can consider that the glider has a past that comes from $-\infty$. The new structure formed by the interaction of the barrier and the glider has now an *infinite* period. The available *bounded* configurations are finite and we know that, since the rule is invertible, something must evolve beyond any bound. We could imagine that the system will grow like an explosion in which the individual tokens separate forever. But, if this were to happen, since the number of tokens is fixed, there will be a finite time at which the distances between them do not allow any interaction. They will end up oscillating in their places in a bigger but still bounded region in contradiction with what we know must happen. Thus, the tokens must evolve in a certain way to create glider(s) that will travel to $\pm \infty$ in the y direction.

Finally, suppose the lattice is infinite in both x and y directions. Again, we have a stable barrier in the x direction and shoot a glider. The structure can remain inside bounds in y forever since all changes can happen in the infinite horizontal direction. It can shoot a glider, it can move, it can do anything.

Now that we have some insight about what can and cannot happen we look at some simple examples. In Figure 9a the glider generates a local perturbation and after approximately a hundred time steps it bounces back leaving behind the barrier unperturbed. We can think of it as a ball bouncing on an elastic membrane. In Figure 9b, however, the perturbation extends to the whole structure and two gliders are ejected in opposite directions with a horizontal displacement relative to the position of the shot. A unit glider with single ballast heading South and a unit glider heading North on the other side. The barrier is left in complete disorder. We stopped our calculations at these time steps in order to show the first part of the evolution, however, these systems have a long life and can generate more interesting behavior. It would be very difficult to predict the outcome of any of these experiments.

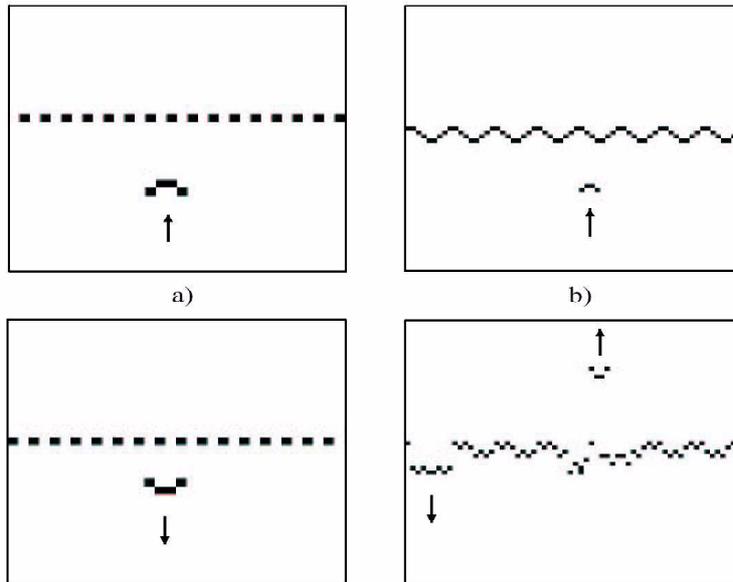


Figure 9 - The upper part of the figures shows the initial set up at time 0. (a) The collision of a glider-seed with a simple barrier in a lattice 32 units wide. In the lower part is the state after 110 steps. In (b) we use a slightly different barrier in a lattice 64 units wide. The result of the experiment after 140 time steps is shown in the lower part.

The barrier used in Figure 9a is stable since each token is in a single block all the time. It is an oscillator of period 4 and remains bounded in the perpendicular

direction by 2 lattice rows. The one of Figure 9b has period 8 and it is bounded in the vertical direction by 4 horizontal lines.

It is interesting to notice that the particles in 2.2 were found in the same way a physicist finds new particles in the real world, i.e., making the known particles collide with each other and with obstacles of different kind as in the above experiments. In *Fermilab* some physicists may wish the quantum world to be so *simple* that particles can be created as in Critters' world, however, their rules seem to have many different *flavors* and to be much more involved.

2.5 Stable and unstable barriers

We have seen some experiments with barriers built 'by hand' and, because of that, what happened in those examples was a solely consequence of the shooting. Now, we would like to set up more general experiments in which the barrier is a bounded horizontal region that goes all around the torus and is composed of a random distribution of tokens with a certain density. When tokens are distributed at random, as in Figure 3, the region is in general not stable; then, if the barrier is thick and with high density and something comes out we can not tell if it was because of the shooting or because of its internal evolution. On the other hand, if the barrier is too thin and with very low density then, the particle may not notice it and pass through it without interacting.

When the barrier is of width 1, i.e., a single line of cells, there are only a few configurations in which it is unstable. We have seen in 2.1 that a long line of tokens forms an oscillator that has a certain period depending on the number of tokens that compose the structure. That is true only when the number of tokens is *finite*; when we place an *infinite* number in a solid line (or close the ring), the structure moves at the speed of light upward or downward depending on the initial position with respect to the blocks. Thus, this is not a stable barrier since it does not remain bounded in the normal direction. It is one of the fastest structures we can find. Moreover, we obtain the same behavior if we make an infinite *dashed* barrier like the oscillators of Figure 4.

If we take out only one token of the solid line the behavior is completely different. The rules are local so the part in the middle of the line (far from the place where the token has been removed) does not notice the absence of the token at the beginning and starts traveling like the whole line does. However, the place where the token is missing has a lack of symmetry with respect to the rest of the line. The tokens in the edges are in separate blocks and the rule for them depends on the time phase. In this case, the barrier splits in two: one formed by a dashed line that travels at the speed of light and another composed of the rest of tokens that behaves like a string. If we take out two tokens of a solid line from the same block, the barrier is also unstable. But these are the only cases (and combinations of them) in which the barrier is unstable. In the majority of cases the barrier is stable and more for

lower densities and larger lattices. Thus, given that the average density $\bar{\delta}$ is small enough and the size of the lattice in the horizontal direction X is large enough the probability of these cases to occur is negligible. Moreover, the probability of the barrier to spontaneously emit a glider of the kind shown in Figure 5, Figure 6 and Figure 7 is zero.

In general, if we distribute the tokens at random in the line, the structure will consist of small strings that oscillate back and forth but remain bounded in the vertical direction. The larger bounds are obtained when the whole line is a string and on its maximum extension it reaches half the size of the lattice, i.e., $X/2$. This is a lower bound for the distance at which we can position a glider to be sure that it starts the flight outside the region of possible conflict.

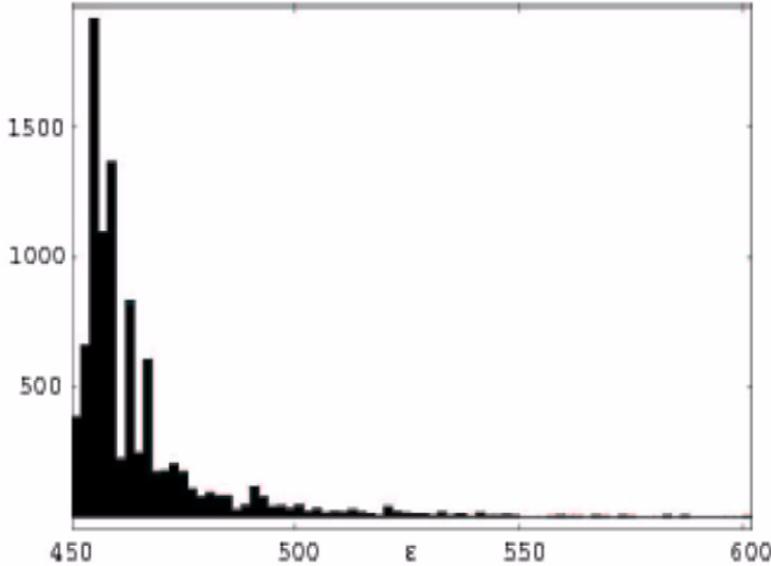


Figure 10 -Frequency distribution of the emission time ε after a shot for a thin barrier with average density $\bar{\delta} = 0.5$. The minimum and maximum times are 434 and 6202, respectively. The mean emission time is $\bar{\varepsilon} = 476.08 \pm 1.21$. The number of experiments is 10000.

2.6 The ‘tunneling’ experiment

In the next experiments we work with the glider of Figure 5 in a lattice of size $X = 128$ units in the horizontal direction and $Y = 256$ units in the vertical direction with periodic boundary conditions. The barrier is a line of width one in which the tokens are distributed at random with a certain probability given by the average density $\bar{\delta}$. The distance from the *cannon* to the *target* is fixed to 100 lattice units³.

³These experiments are run in the lattice gas model so a lattice unit is the separation between processors of the mesh and corresponds to two cells in the partitioning scheme.

We place a *detector* at a distance of 128 units from the barrier. This *device* simply scans a horizontal line every time step and register the time when something hits. At this moment the lattice is cleaned and another experiment starts. We measure the time at which the *first* particle is detected and also from which side of the barrier it came out.

The time we measure has actually three components, namely, the time it takes for the glider to reach the boundaries of the barrier, the time it spends interacting with the barrier and the time it takes to reach the detector. But for all the experiments the first and the third times are approximately equal, so the results express the amount of time the glider interacts with the barrier.

Figure 10 shows the frequency distribution of the emission time ε for 10000 experiments with a barrier of average density $\bar{\delta} = 0.5$. The mode is 454 with 1358 cases and corresponds to a bouncing glider. There are a lot of peaks alternating with valleys of different magnitude. For example, near the mode there is a deep valley in which the emission time of 453 has only 7 cases.

We performed the same number of experiments (10000) with barriers of different densities. The gliders will come on the upper side (through the barrier) or on the lower side (bouncing) with a certain probability which is a function of the average density. The percentage of bouncing gliders which is zero for zero density grows to 50% for a density of about 0.15 then it continues growing to reach 88% at $\bar{\delta} = 0.5$. In the experiments shown in Figure 10 the number of gliders that were captured on the side of the shot (bouncing) was 8808. For greater densities, the percentage of bouncing gliders, remains constant and for densities of 0.7 and 0.8 it has a slight decrease.

The mean emission time $\bar{\varepsilon}$ grows slowly when the average density is below 0.8 with a small standard error. For $\bar{\delta} = 0.3$, $\bar{\varepsilon} = 458.30 \pm 0.13$ and the minimum and maximum times were 444 and 1216 time steps, respectively. For $\bar{\delta} = 0.7$, $\bar{\varepsilon} = 733.7 \pm 25.21$ and the minimum and maximum times were 427 and 165170 steps, respectively. When we increase the average density, the barrier is composed of long strings intertwined along the horizontal direction with longer periods. When the glider gets near these strings the system enters a long sequence of changes which are slight modifications of the actual period of the barrier until it is expelled out. For $\bar{\delta} = 0.8$, $\bar{\varepsilon} = 3242.96 \pm 399.02$ and the minimum and maximum times were 388 and 2597864 steps, respectively. For densities above 0.8 the time of the experiments becomes too long for the above mentioned set up and there is a larger probability of having unstable barriers of the dashed type, like the one discussed in 2.5.

We have only looked at the time of the first particle coming out but not at the kind of particle coming out. In most of the experiments, a typical glider (see Figure 5) is expelled. However, there is a certain probability of having different kinds of particles coming out (as in Figure 9b), especially when the barrier becomes more

dense and involved.

3 Macroscopic dynamics

It would be very interesting to extend the above experiments to thicker barriers but, as we already mentioned, if we use a thick barrier we have uncertainties about its actual stability. In the previous case, when the barrier was of width 1 we could get rid of the problem by choosing a large horizontal lattice. But when two or more lines are filled at random, even for small densities, there is a non-zero probability that a glider of the ones in Figure 5, Figure 6 or Figure 7 is spontaneously generated. This was the main reason to study the stability of the system as a function of the density, that is, a study of the diffusion properties of the rule. In this part we show the new kind of experiments and the results that attempt to answer some of these questions.

3.1 Critters and diffusion

If we drop ink in a glass of water the molecules of ink will spread through the whole container according to a diffusion process. Molecules of water and ink will ‘exchange places’ randomly while hitting each other performing a *brownian motion*. The concentration of ink will decrease with the distance from the center of the initial spot proportionally to the square root of time. The system will increase its disorder and, in the end, we get a uniform mixture of water and ink. This process is governed by the diffusion equation,

$$\frac{\partial \delta}{\partial t} = D \frac{\partial^2 \delta}{\partial x^2} \quad (1)$$

where $\delta(x, t)$ is the concentration of the substance along the x direction as a function of time t . The constant D is the *diffusion coefficient* or *diffusivity*, and it depends on the substance.

If Critters’ rule were linear, we could completely characterize its macroscopic behavior with Eq. (1) by finding its diffusion constant. However, it is easy to see that, in Critters, there is a strong dependence of the diffusivity on the concentration of tokens. We know that, if the density is too low, i.e., tokens are sufficiently isolated, there is no diffusion at all since all the tokens remain in a 4 cycle. Also, since the rule is symmetric with respect to ‘ones’ and ‘zeros’ the same happens when the density of tokens is a maximum. Only when the concentration is between these limits do the tokens begin to interact with each other and diffuse.

In order to find the dependence of the diffusivity on the density we establish a difference of concentration and measure how fast the system goes to equilibrium. We can reduce the problem to one-dimension simplifying the experiments, i.e., we work as if the system were a ‘tube’ in which the differences in density are along its axis in the horizontal direction x . The vertical direction y (circumference of the tube)

allows to increase the amount of samples at each point. Also, in order to have no boundary effects we use a lattice with periodic boundary conditions and a smooth, periodic sinusoidal distribution for the density as a function of x . Figure 11 shows the basic set up.

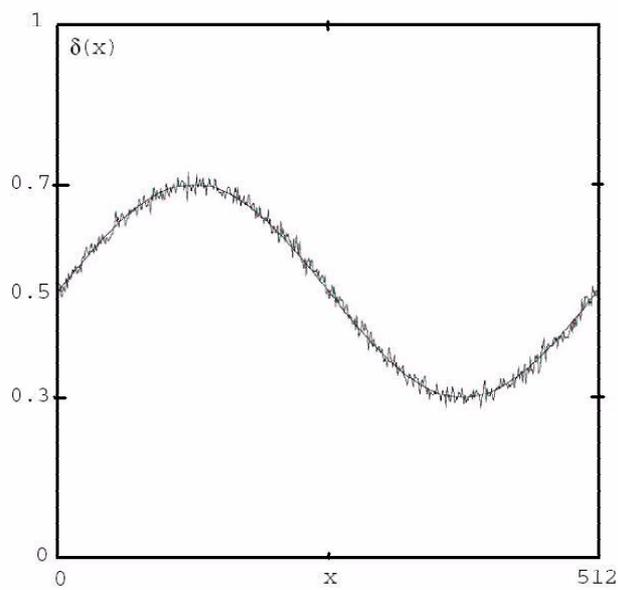
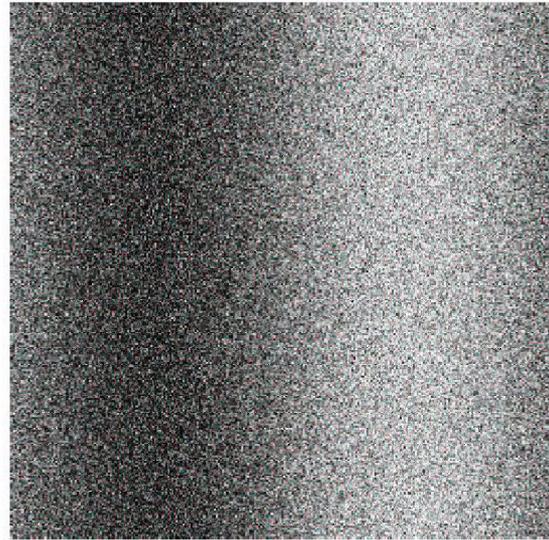


Figure 11 - In the upper part of the figure is the distribution of signals in a mesh with 512×512 processors ($X = 512$) at the beginning of a typical experiment. Darker zones imply higher density and vice-versa. In the lower part is the concentration given by Eq. (2) together with the actual density on each column. The mean density is $\bar{\delta} = 0.5$, the spatial

frequency is $\omega = 2\pi/X$ and the initial amplitude $a_0 = 0.2$.

The function that represents the decay process is given by,

$$\delta(x, t) = a(t)\sin(\omega x) + \bar{\delta} \quad (2)$$

where $\bar{\delta}$ is the average density and ω is the frequency of the wave. It is easy to check Eq. (2) satisfies Eq. (1) if $a(t) = a_0 e^{-D\omega^2 t}$. Where a_0 is the maximum concentration at time 0.

In each experiment we initialize the signals at random with a probability given by Eq. (2) at time $t = 0$; this is our peculiar spot of ink. Then we let the program run and measure the concentration along the tube at different times. With the data obtained in this way we perform a surface fitting to find the values of three parameters: initial amplitude, average density and diffusivity⁴.

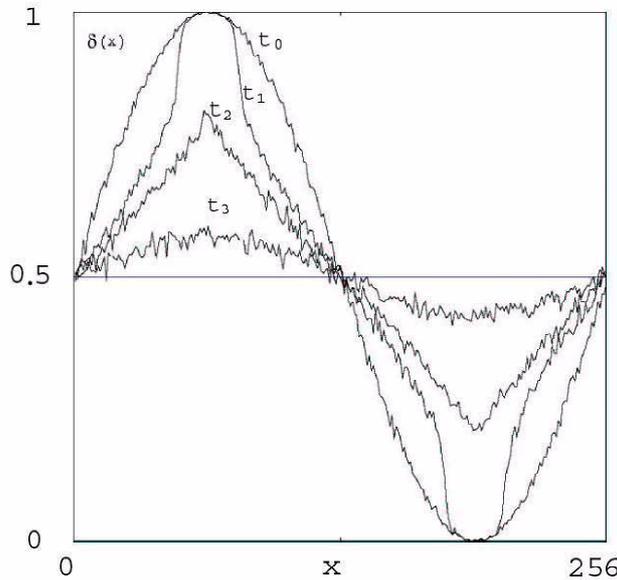


Figure 12 - The decay process of the density δ along the tube for four different times. At t_0 we distribute the tokens at random with a probability given by Eq. (2) for $X = 256$, $Y = 512$, $\bar{\delta} = 0.5$ and $a_0 = 0.5$. The time t_1 is after 200 time steps, the time t_2 after 500 steps and t_3 after 1000 time steps.

In Figure 12 we see four sections of the surface of density. We use the maximum possible amplitude to show the nonlinear behavior of the decay process. The places where the density is a maximum (or minimum) do not decay, that is, there is

⁴Because of the symmetry of the rule and the periodicity of the function and boundary conditions we consider the frequency ω not susceptible of variations due to noise, i.e., is considered a constant during the time of the experiment.

no diffusion at the beginning. The only place where diffusion takes place from the beginning is where the density is near 0.5. After 1000 steps the density seems to return to a sinusoidal and the process continues in this way until the curve flattens at maximum entropy for a uniform average density of $\bar{\delta} = 0.5$. In this case we cannot assume that the system behaves like a diffusing process but if we start the experiment with a sufficiently small amplitude a_0 we can have a good approximation.

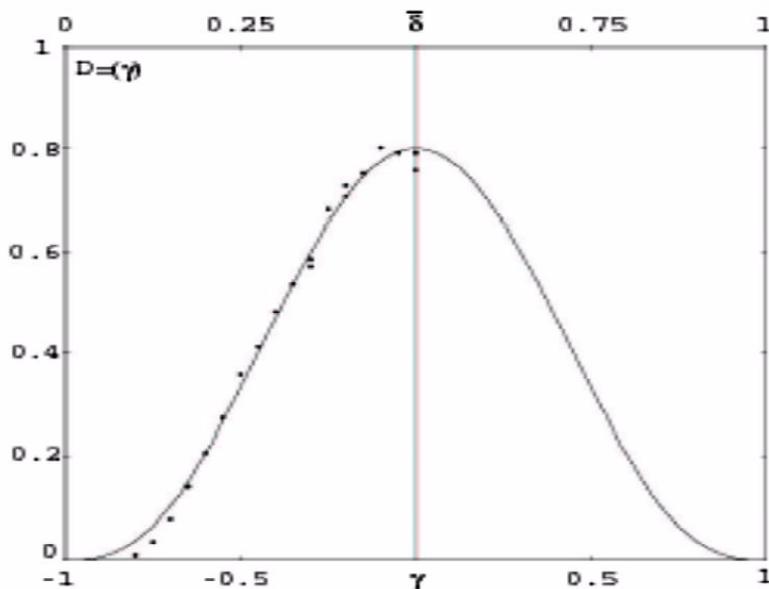


Figure 13 - Experimental results (points) and fitting (solid line) given by Eq. (8), of the diffusivity D as a function of the mean density $\bar{\delta}$ (upper horizontal axis). Since the curve is symmetric with respect to $\bar{\delta} = 0.5$ we scale the average concentration using $\gamma = 2\bar{\delta} - 1$ (lower horizontal axis).

When the average density $\bar{\delta}$ is too low or too high, the upper and lower parts of the wave decay at different rates because the difference in density implies a difference in diffusivity. Also, in these cases the decay rate is so small that, to obtain an appreciable difference with respect to the initial distribution, the running time had to be increased. The combination of these factors allowed us to study the diffusivity only in the interval $0.1 \leq \bar{\delta} \leq 0.5$.

Figure 13 shows the dependence of D on the mean density $\bar{\delta}$. The experimental results (points) are shown for half the spectrum of densities since the curve is symmetric with respect to $\bar{\delta} = 0.5$. In order to fit the experiments with the bell shaped curve shown in the figure we looked for a simplification of the process and a theoretical model that can explain this behavior in terms of a random walk. The function will be obtained in the next section.

As pointed out at the beginning of this section, the diffusivity goes to zero in the limits of very low or very high densities and has a maximum when the concentration of tokens is $1/2$. Each point in Figure 13 is the average of 10 experiments.

We have also studied the dependence of the diffusivity with the frequency in Eq.(2) for fixed $\bar{\delta} = 0.5$ and $a_0 = 0.02$. It is easier to interpret the results in terms of wavelengths. When the wavelength is greater than 16 lattice units the diffusivity remains around 0.8 with smaller dispersion for larger wavelengths. When the wavelength is smaller than 16 lattice units, D increases to reach approximately 1.4 for wavelengths of the order of 6 lattice units with large dispersion. Finally, when the wavelength becomes of the order of magnitude of the units of the mesh it is impossible to measure D with any accuracy.

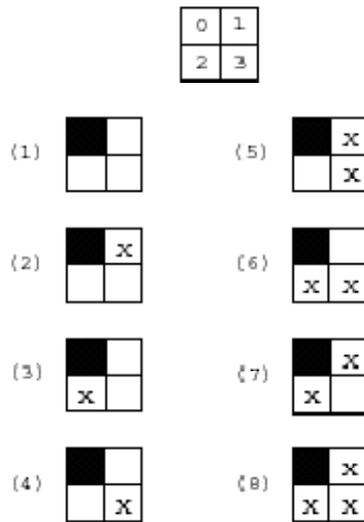


Figure 14 - In the upper part is a general block with indexed positions and in the lower part, the eight different cases in which a token at 0 can be for different amount and positions of neighbors (represented with a cross).

3.2 The random walker approach

Now we come back to the microscopic partitioning scheme which allows us to make a first approximation, much like a *Boltzmann approximation*, in which we consider the neighbors of a token to be uniformly distributed and make a statistical analysis. If we look at a particular token inside a block at any given step it can move to 4 different places; the probability of a movement to one or the other depending on the amount and position of its neighbors. There are 8 different cases, shown in Figure 14, each one with the following probability of occurrence,

$$\begin{aligned}
 p_1 &= (1 - \bar{\delta})^3 \\
 p_2 &= p_3 = p_4 = \bar{\delta}(1 - \bar{\delta})^2 \\
 p_5 &= p_6 = p_7 = \bar{\delta}^2(1 - \bar{\delta}) \\
 p_8 &= \bar{\delta}^3
 \end{aligned}
 \tag{3}$$

Given that there is a token in the block and that the neighbors are placed at random with a certain probability given by the mean density we ask: which is the probability that the token goes to one of the eight nearest neighbors in a 3x3 square and which is the probability of its remaining in place?

By means of the rule given in Figure 1 we can find the probabilities that, at the next step, the given token is in one of the four positions inside the block; they are represented by q_0 , q_1 , q_2 and q_3 . There is certain freedom in the election of these four probabilities since for example, when two cells of a block are occupied as in case (2) or (3) of Figure 14 we can assume that each of the tokens has the same probability to go to any of the two result cells or we can assume they cross to opposite corners with probability one, etc. We use a model in which when the token is alone, case (1) in Figure 14, or with the other three neighboring sites filled, case (8), it remains in place with probability one. In the other cases we simply apportion the probabilities among the possible outcomes. The assignments for the odd rule of Figure 1 are given by the following equations,

$$\begin{aligned}
 q_0 &= p_1 + 1/3(p_5 + p_6) + p_8 \\
 q_1 &= 1/2(p_3 + p_4) + 1/3 p_6 \\
 q_2 &= 1/2(p_2 + p_4) + 1/3 p_5 \\
 q_3 &= 1/2(p_2 + p_3) + 1/3(p_5 + p_6) + p_7
 \end{aligned}
 \tag{4}$$

For example, q_0 is the probability of the token to remain in place and it is the sum of the probabilities of (1), (8) and the third of the probabilities on (5) and (6) since it is possible for it to remain in place in these cases, too. With these probabilities we have the following transition-probability matrix,

$$\begin{pmatrix}
 q_0 & q_1 & q_2 & q_3 \\
 q_2 & q_0 & q_3 & q_1 \\
 q_1 & q_3 & q_0 & q_2 \\
 q_3 & q_2 & q_1 & q_0
 \end{pmatrix}
 \tag{5}$$

which because of rotational symmetry and the chosen assignments is double stochastic, i.e., the sum of each column and each row is one, and represents an irreducible Markov process, i.e., all states communicate and the probability of a transition to any state depends only on the previous visited state and not in the whole history of transitions. Now, we are ready to find the probabilities of a movement on each direction in a 3x3 square. We assume that we have the same probability for any of the four possible 2x2 blocks inside the square with the cell in the center occupied.

Figure 15 shows the ‘octopus walker’ with probabilities in each direction given by,

$$\begin{aligned} p_N &= p_S = p_E = p_W = 1/4(q_1 + q_2) \\ p_{NE} &= p_{NW} = p_{SE} = p_{SW} = 1/4 q_3 \\ p_C &= 1 - 4(p_N + p_{NE}) \end{aligned} \quad (6)$$

where p_C is the probability of remaining in place.

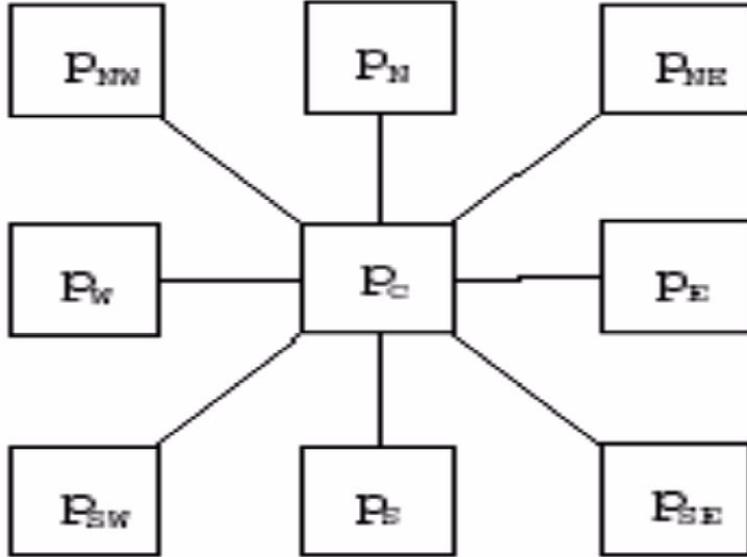


Figure 15 - The ‘octopus random walker’ with weighted arms.

A measure of the diffusivity of this random walker is given by the *moment of inertia* of the weights in its *arms* with respect to a perpendicular axis passing through the center cell, that is, the sum of the products of the square of the distances and the masses. Taking into account that in the diagonal direction the distance traveled by the token is $\sqrt{2}$ the moment gives,

$$\mu = 8p_N + 4p_{NE} = 4\bar{\delta}(1 - \bar{\delta}) \quad (7)$$

where we have written p_N and p_{NE} in terms of $\bar{\delta}$ using Eq. (3), (4) and (6). As expected, the function goes to zero when $\bar{\delta} = 0$ and $\bar{\delta} = 1$ and has a maximum at $\bar{\delta} = 0.5$. However, the curvature of the parabola is completely different from that of the experiments due to the *simplistic* assumption that the neighbors are distributed at random. In the real dynamics there are *second-order* and probably *n-order* correlations which we simply neglected. The assignments for the even part of the rule were not taken into account, either. In spite of all these simplifications it can be considered a good starting point to build a better fit. We assume that all

the differences with respect to the experiments can be gathered in two parameters r and s introduced in the following way,

$$D(\bar{\delta}) = r(4\bar{\delta}(1 - \bar{\delta}))^s \quad (8)$$

and we find the values of r and s that best fit the experiments in Figure 13. With $r = 0.799 \pm 0.007$ and $s = 3.037 \pm 0.086$ we have a good approximation for the range $0.175 \leq \bar{\delta} \leq 0.5$.

To stress the fact that the curve is symmetric with respect to $\bar{\delta} = 0.5$ we rescale the function using $\gamma = 2\bar{\delta} - 1$ to have,

$$D(\gamma) = r(1 - \gamma^2)^s \quad (9)$$

where now the independent variable γ ranges from -1 to 1.

The high dependence of the diffusivity on the mean density shows how far from linearity is Critters' rule. It should be noticed that these results are valid when the gradient of concentration between two nearby points of the lattice is small. When the change in concentration is abrupt in the boundaries of a region, like in Figure 3, the behavior is even more complex and other models should be built to analyze them. This is the case for the study of stability and tunneling with thick barriers.

4 Conclusions

We have presented microscopic and macroscopic characteristics of the cellular automaton called Critters. We studied the kind of *oscillators* and *particles* we can construct. We also made them collide with each other and with barriers of different kinds.

We studied the distribution of time after an emission for a thin barrier with average density equal to the maximum entropy and the *mean emission time* for different densities up to 0.8.

We made experiments to capture the diffusion properties of the rule and found the curve of diffusivity as a function of density. Despite the complexity of the rule we presented a possible explanation based on the movement of a given 'token' or 'signal' as a random walker. To build more accurate models it would be necessary to take into account higher order correlations between tokens.

We followed just a few branches of a big tree and many of the paths we took have been explored only on their surface. There is still a lot to do if we want to make more general *tunneling experiments* and get more useful information about this phenomenon. However, the experiments shown here and the general treatment

can be a good starting point for further investigation.

Finally, there are many other cellular automata like Critters that have never been seen running in the screen of a computer and which are waiting to be explored. Some of them may be more *physical* than others, some of them may be more *linear* than others but all show different universes inhabited by strange *cultures* that are worth to explore as are the ones in our own world.

5 Acknowledgements

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