Prediction of the mean path of random walkers

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Abstract: *A random walker is allowed to walk on a surface along two orthogonal directions; the possibility to select either of them is a specific function of the number of steps he has performed. Once he has chosen one of the two axis he is also given a possibility to choose a forward or a backward step; this possibility may also be a specific function of the number of steps. In spite of the wide variety of possibilities that may be assigned we herein describe a theoretical method through which the mean path of a large number of walkers may be found. Several numerical examples for random walkers verify the theoretical results for their mean path.*

Key words: *Random walkers, simulation of continuous functions with discrete steps, Method of Expectancies, expectation of random variables*

Resumo: *Permite-se que um caminhante percorra uma trajetória aleatória sobre uma superfície ao longo de duas direções ortogonais; a possibilidade de selecão entre elas é uma função específica do número de passos que ele executa. Após escolher um dos dois eixos, também lhe é permitido dar um passo para frente ou para trás; esta possibilidade também pode ser uma função específica do número de passos. Apesar da grande variedade de possibilidades que se pode escolher, descrevemos um método teórico pelo qual se pode encontrar o caminho médio de um número grande de caminhantes. Vários exemplos numéricos verificam os resultados teóricos para os caminhos médios.*

Palavras-chave: *caminhadas aleatórias, simulação de funções contínuas com passos discretos, Método da Expectativa, expectativa de variáveis aleatórias*

1 Introduction

The study of Brownian motion was one of the first successful achievements of random walkers. Afterwards the theme has evolved and grown in several areas of science such as physics, molecular biology, chemotaxis and mathematics [1-11]. The origin of the theory of the present paper and its applications (growth of objects such as tumors, cities, fluid motion in porous media, etc.,) may be found in Refs. [12-18]. For example, in Ref. [14] the method presented in this article has been successfully applied to the study of the area-perimeter relationship of objects grown with models of aggregation phenomena.

2 Theory

Let us consider a plane where a random walker roams; his position is defined by orthogonal axes X, Y along the horizontal and vertical directions, respectively. The walker has only two classes of steps: He may choose either a step along the X-axis, with possibility $P(n)$, or select a step along the Y-axis with possibility $1 - P(n)$.

The theoretical basis of this paper is rather simple. Let us concentrate our attention, for the moment, on his steps along the X-axis. We assume that a walker makes n^+ positive steps of unit length along the X-axis; he may also make n^- steps in the opposite (negative) direction. After a certain number of steps, $n = n^+ + n^-$, the distance reached by the walker is $\Delta X = n^+ - n^-$. If we denote with $p(n) = p(n^+)$ and $p(n^-) = 1 - p(n)$ the probabilities of positive and negative steps, respectively, then the ratio

$$\frac{dX}{dn} = \lim_{W\to\infty} \frac{1}{W} \sum_{i=1}^{W} \frac{n_i^+ - n_i^-}{n_i^+ + n_i^-} = 2p(n) - 1 \tag{1}$$

tends to the derivative (that may be herein called 'probabilistic' derivative) of the mean distance reached by a large number W of walkers with respect to the number n of steps. The path of a random walker is discrete, composed of steps of unit length. However, as we will see in the following paragraphs, it is convenient to consider the average path of a large number of walkers as a continuous function.

If we follow the same reasoning for a walker making steps along the Y-axis we arrive at the following system of differential equations for the mean path of a large number of random walkers on a plane

$$\begin{aligned} \frac{dY}{dn} &= [1 - P(n)]\,[2p_y(n) - 1] = F(n) \\ \frac{dX}{dn} &= P(n)\,[2p_x(n) - 1] = G(n) \end{aligned} \tag{2}$$

We have introduced the sub-indexes x and y to denote different possibilities of positive steps along X and Y-axis, respectively. We have herein assumed independence between $P(n)$, $p_x(n)$ and $p_y(n)$.

Equations (2) define a so-called *nonautonomous* system in which X and Y do explicitly depend on 'time' n, i.e. $F(n)$ and $G(n)$. For some examples of *autonomous* systems, i.e. $F(X, Y)$ and $G(X, Y)$, see Ref. [12]. The generalization of Eqs. (2) to a larger number of variables, not herein shown, is immediate, though the solution may be rather involved.

In the following paragraphs we will study different cases of interest. All walkers start their journey at the point of coordinates (X_0, Y_0) for $n = 0$. A few representative examples, with different functions for the possibilities $P(n)$, $p_x(n)$ and $p_y(n)$ are selected in order to see the pattern of the walk and to test the theory for the equation of the mean path. A measure of the linear dimensions of the example is a square frame of sides equal to L units of length.

2.1 Example 1

We will start our study with a trivial random walker. Figure 1 shows $W = 1500$ walkers, all of them starting their paths at the lower left corner of the frame, and ending it when they have performed $ns = 7000$ steps, selecting the X-axis with a constant possibility $P(n) = 0.55$. They also have constant possibilities $p_x(n) = 0.6$ and $p_y(n) = 0.6$ to go forward once they have selected the respective axis. It is obvious to say that the possibilities to choose the Y-axis and a backward step are $1 - P(n) = 0.45$ and $1 - p_x(n) = 1 - p_y(n) = 0.4$, respectively. The 1500 paths show the classical dispersion of random walkers, proportional to $\sqrt{n}$ when n increases.

By means of Eqs. (2) we may find the equation of the mean path

$$
\begin{aligned}
Y - Y_0 &= \int_0^n [1 - P(n)]\,[2p_y(n) - 1]\,dn = 0.09n \\
X - X_0 &= \int_0^n P(n)\,[2p_x(n) - 1]\,dn = 0.11n
\end{aligned}
\tag{3}
$$

This simple system of parametric equations allows the expression

$$
Y = Y_0 + \frac{0.09}{0.11}\,(X - X_0) \tag{4}
$$

for the mean path. It is painted in white in the figure above; the origin is denoted with a small circle near the lower left corner of the frame.

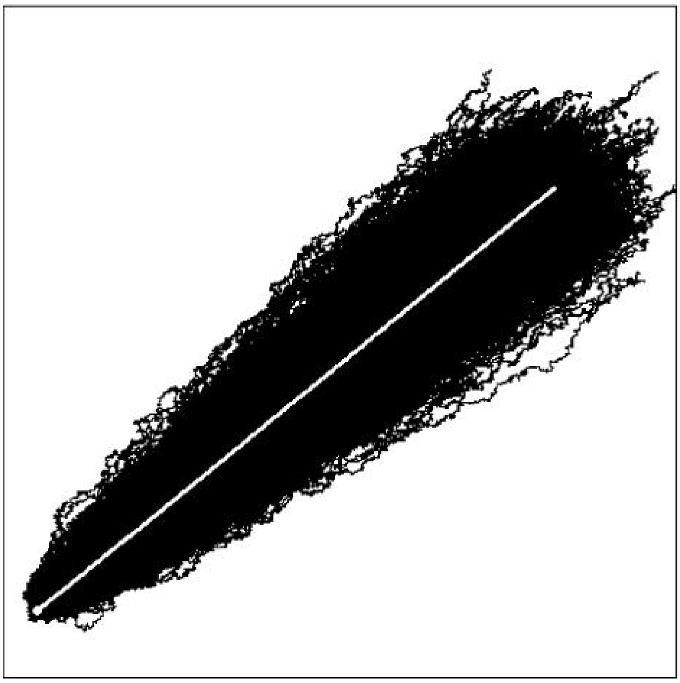

Figure 1. The simplest case, given by Example 1, with constant probabilities to choose a step in each direction. The trajectories of random walkers are painted in black and the mean path, given by Eq. (4), in white.

2.2 Example 2

This case also has $W = 1500$ random walkers, each one with a maximum number of steps of $ns = 7000$, as in Example 1. Figure 2 shows the results enclosed in a square frame of $L = 1650$. The numerical model (experimental results) represented with black paths show a wavy pattern due to the following initial conditions of possibilities

$$\begin{aligned} P(n) &= n/ns \\ p_x(n) &= 0.7 \\ p_y(n) &= 0.6 - 0.4\sin(2\pi n/2000) \end{aligned} \tag{5}$$

The first of these three equations above says that at the start of each walk, when $P(n)$ is near zero, most of the steps are likely to be along the Y-axis; for the final steps, when $P(n)$ is near 1, most of the steps may be expected to be along the X-axis. The second equation tells the walker that he has more possibilities to make a forward step along the X-axis and a few in a backward direction. The third equation gives an oscillatory nature to the possibility of a positive or a negative step along the Y-axis.

In order to know the theoretical equation of the mean path we replace Eqs. (5) into Eqs. (2)

$$\begin{aligned} Y - Y_0 &= \int_0^n [1 - P(n)]\,[2p_y(n) - 1]\,dn = -254.648 + 0.0000571429(3500 - n/4)n \\ &\quad -0.0363783(n - 7000)\cos(\pi n/1000) + 11.5796\sin(\pi n/1000) \qquad (6) \\ X - X_0 &= \int_0^n P(n)\,[2p_x(n) - 1]\,dn = 0.0000285714n^2 \end{aligned}$$

These parametric equations are represented with a white trace. It may be clearly seen that the mean path represents reasonably well the paths of 1500 random walkers.

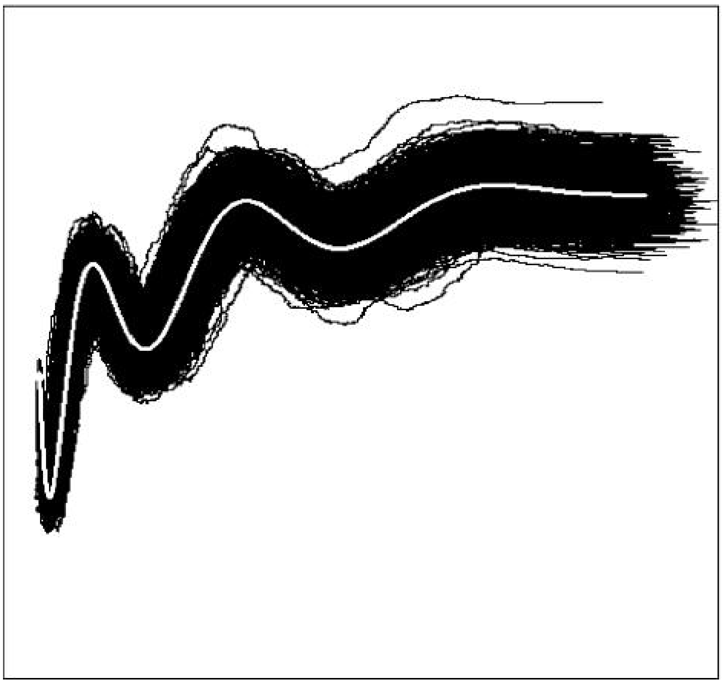

Figure 2. The trajectories of W = 1500 random walkers each one making a total of ns = 7000 steps are shown in black. The probabilities for each direction are given by Eqs. (5) and the mean path, painted in white, is given by Eqs. (6).

2.3 Example 3

This is a case $W = 1000$ of random walkers, each one performing $ns = 20000$ steps on a region limited by $L = 2800$. The possibilities for the election of a positive or a negative step along both axis are trigonometric functions out of phase. The possibilities are

$$\begin{aligned} P(n) &= 0.55 \\ p_x(n) &= 0.4\,[1+\sin(2\pi(n/1250-0.25))] \\ p_y(n) &= 0.4\,[1+\sin(2\pi n/5000)] \end{aligned} \tag{7}$$

Figure 3 shows the 1000 meandering paths painted in black, while the mean path, painted in white, is given by the following equations

$$\begin{aligned} Y-Y_0 &= \int_0^n [1-P(n)]\,[2p_y(n)-1]\,dn = 0.45\,[-0.2n+636.6198\,(1-\cos(\pi n/2500))] \\ X-X_0 &= \int_0^n P(n)\,[2p_x(n)-1]\,dn = 0.55\,[-0.2n-159.15494\cos(2\pi(n/1250-0.25))] \end{aligned} \tag{8}$$

The origin of walks is in the upper right corner of the square.

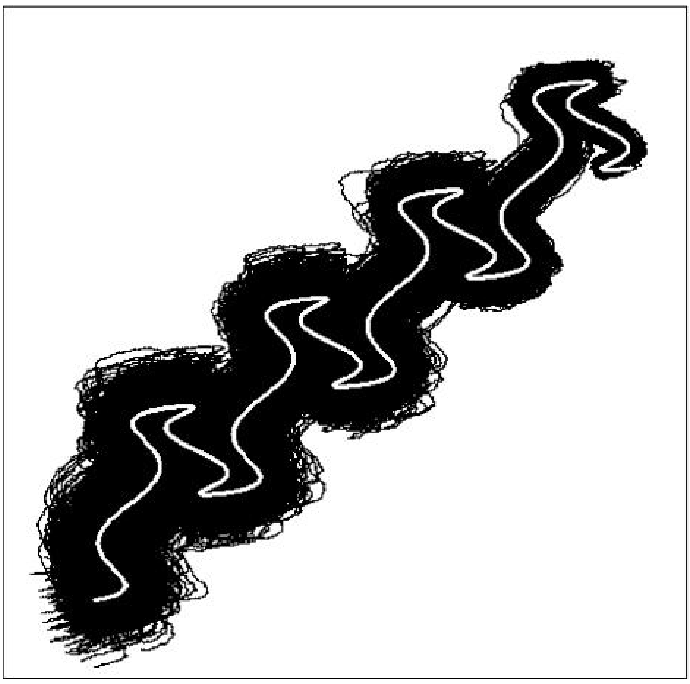

Figure 3. This figure illustrates Example 3 showing the 1000 meandering paths painted in black, while the mean path, painted in white, is given by Eqs. (8). The starting point for all walkers is in the upper right corner of the region.

2.4 Example 4

Figure 4 shows long paths ($ns = 35000$) of $W = 1000$ random walkers that converge with spirals towards the interior of a square of $L = 4200$ units. The possibilities of their random motion are similar to the previous example except for exponential terms

$$\begin{aligned} P(n) &= 0.5 \\ p_x(n) &= 0.5\left[1+\sin\left(2\pi(n/25000-0.25)\right)\exp\left(-3.4\times 10^{-5}n\right)\right] \\ p_y(n) &= 0.5\left[1+\sin\left(2\pi n/25000\right)\exp\left(-3.4\times 10^{-5}n\right)\right] \end{aligned} \tag{9}$$

The integration of Eqs. (2) with Eqs. (9), the mean path is,

$$\begin{aligned} Y - Y_0 &= \int_0^n [1-P(n)]\,[2p_y(n)-1]\,dn = 0.5\,\{53907.36 - 3907.36\exp(-0.000034n) \\ &\quad \times [\cos(\pi n/12500) + 0.135282\sin(\pi n/12500)]\} \\ X - X_0 &= \int_0^n P(n)\,[2p_x(n)-1]\,dn = 0.5\,\{-5280595 + 528.595\exp(-0.000034n) \\ &\quad \times [\cos(\pi n/12500) - 7.391983\sin(\pi n/12500)]\} \end{aligned} \tag{10}$$

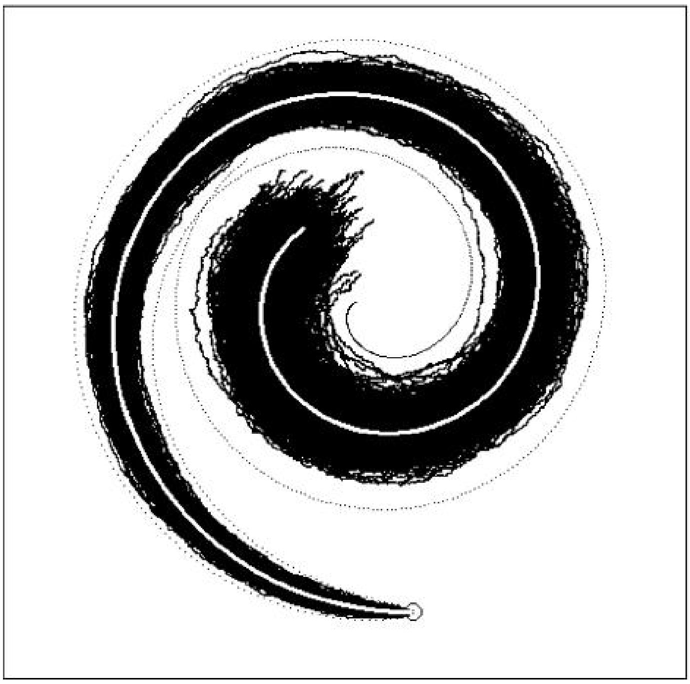

Figure 4. Random walkers with a large amount of steps, ns = 35000, and probabilities given by Eqs. (9). Their mean path, shown in white, is an inward spiral given by Eqs. (10). Two dotted lines at a distance of 3 $\sqrt{n}$ normal to the mean path embrace most of the trajectories.

The converging spirals of this example (starting at the lower part of the frame) can be transformed into diverging ones by changing the sign of the argument of the exponential from negative to positive.

2.5 Example 5

In figure 5 there are $W = 500$ walkers (with black paths), each one with $ns = 85000$ steps, in this example. They are instructed with the following possibilities

$$\begin{aligned} P(n) &= 0.3 \\ p_x(n) &= 0.6 + 0.4 \left[\sin\left(2\pi(n/25000 - 0.25)\right) \exp\left(-3.4 \times 10^{-5} n\right)\right] \\ p_y(n) &= 0.5 + 0.22 \left[\sin\left(2\pi n/25000\right) \exp\left(10^{-5} n\right)\right] \end{aligned} \tag{11}$$

The main difference between this and the previous example is that the amplitudes of $p_x(n)$ decrease, and those of $p_y(n)$ increase, with n. The result is that they walk from left to right with increasing amplitudes.

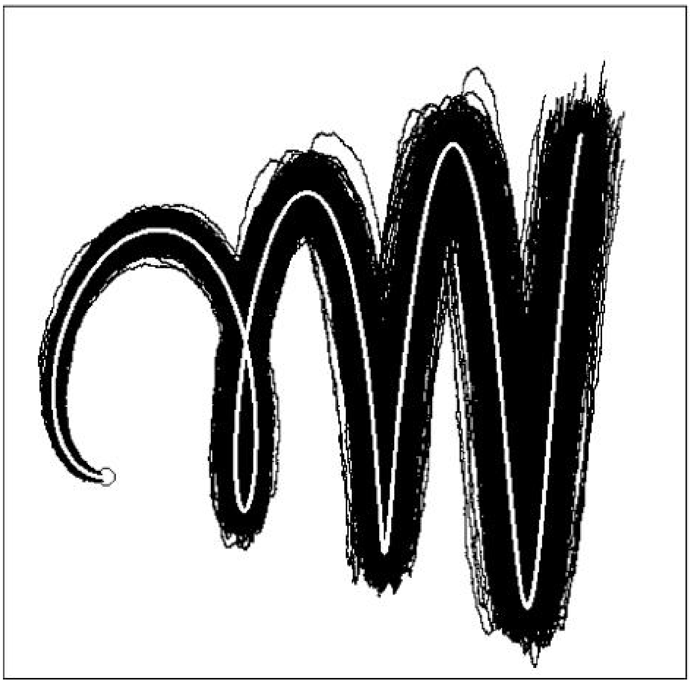

Figure 5. Despite the complicated probabilities given by Eqs. (11), the mean path, in white, follows extremely well the trajectories of these random walkers. A white circle on the left shows the starting point of all walks.

The region is limited by the frame of $L = 7100$. The mean path is given by the equations

$$\begin{aligned}
Y - Y_0 &= \int_0^n [1 - P(n)][2p_y(n) - 1]dn = 0.7\left\{1747.937 - 1747.937\exp\left(10^{-5}n\right)\right. \\
&\quad \left.\times\,[\cos(\pi n/12500) - 0.0398873\sin(\pi n/12500)]\right\} \\
X - X_0 &= \int_0^n P(n)[2p_x(n) - 1]dn = 0.3\{-422.875 + 0.2\exp\left(-3.4\times 10^{-5}n\right) \\
&\quad \times[n\exp\left(3.4\times 10^{-5}n\right) + 2114.38\cos(\pi n/12500) \\
&\quad -\, 15629.457\sin(\pi n/12500)]\}
\end{aligned} \tag{12}$$

2.6 Example 6

Figure 6 shows a case in a field of $L = 1700$ in which $W = 1500$ walkers make $ns = 12000$ steps. We want that almost all steps should be along the Y-axis at the beginning of the walk; thus we set a small $P(n)$ for small n. However, we also want that all steps should be along the X-axis when the path is about to end; this is accomplished with a high $P(n)$ when n approaches ns. The possibilities are given by

$$\begin{aligned}
P(n) &= 0.2 + 0.8\,n/ns \\
p_x(n) &= 0.5 + 0.5\sin(2\pi n/3000) \\
p_y(n) &= 0.6 + 0.4\sin(2\pi n/3000)
\end{aligned} \tag{13}$$

Figure 6. This figure shows the results for Example 6. The black paths show the behavior of 1500 random walkers and the white curve shows their mean path.

The 1500 random walkers make the paths shown in black; the mean path (in white) is found with the integration of Eqs. (2) with Eqs. (13)

$$
\begin{aligned}
Y - Y_0 &= \int_0^n [1 - P(n)]\,[2p_y(n) - 1]\,dn = 305.577 + 0.00002667\,(6000 - 0.25n) + \\
&\quad 0.0254648\,(-12000 + n)\cos(\pi n/1500) - 12.1564\sin(\pi n/1500) \qquad (14) \\
X - X_0 &= \int_0^n P(n)\,[2p_x(n) - 1]\,dn = 95.49297 - 0.031831\,(3000 + n)\cos(\pi n/1500) + \\
&\quad 15.198177\sin(\pi n/1500)
\end{aligned}
$$

2.7 Example 7

A peculiar path resembling a number 8. Figure 7 shows $W = 1000$ random walkers each one with $ns = 25000$ steps contained in a square of $L = 5100$. They must obey the possibilities

$$
\begin{aligned}
P(n) &= 0.5 \\
p_x(n) &= 0.5 + 0.5\sin[2\pi\,(n/12500 - 0.25)] \qquad (15) \\
p_y(n) &= 0.5 + 0.5\sin(2\pi n/25000)
\end{aligned}
$$

Their mean path (in white) is obtained with the above possibilities and the integration of Eqs. (2)

$$
\begin{aligned}
Y - Y_0 &= \int_0^n [1 - P(n)]\,[2p_y(n) - 1]\,dn = 0.5\,[3978.87 - 3978.87\cos(\pi n/12500)] \\
&\qquad (16) \\
X - X_0 &= \int_0^n P(n)\,[2p_x(n) - 1]\,dn = 0.5\,[-1989.4368\cos(2\pi(n/12500 - 0.25))]
\end{aligned}
$$

The paths start at the lower part of the figure (white circle); the walkers are due West at the beginning of their journey. The number of steps has been selected in order to end the path near the origin (X_0, Y_0); they come from the East. Had the number of steps been considerably increased the number 8 would disappear on account of the dispersion on the random walks. However, the number 8 for the mean path will remain unaltered.

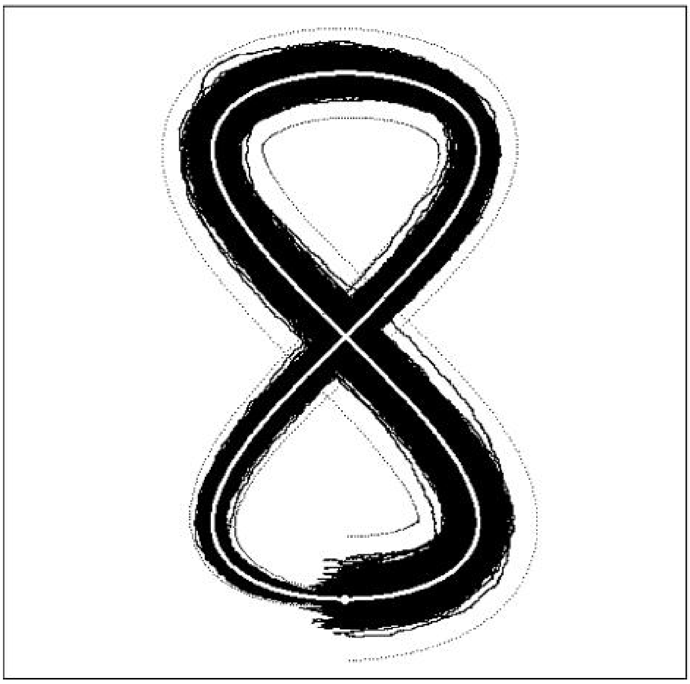

Figure 7. The trajectories of random walkers following directions given by Eqs. (15) are painted in black. The mean path, in white, is a periodic curve, a figure-eight given by Eqs. (16). The dotted lines at a distance of 3 $\sqrt{2}$ normal to the mean path embrace the wake left by the walkers.

3 Dispersion of walks

When each of the walkers completes its ns steps they reach a point $(X_{ns,i}, Y_{ns,i})$, for $1 \leq i \leq W$; the final point of the mean path is (X_{ns}, Y_{ns}). The quantity

$$\varepsilon = \frac{1}{W} \sum_{i=1}^{W} \sqrt{(X_{ns,i} - X_{ns})^2 + (Y_{ns,i} - Y_{ns})^2} \tag{17}$$

is a measure of how dispersed the walkers are when they end their traveling. It seems evident that $\varepsilon \to 0$ when $W \to \infty$. We have used reasonable values of W in order to have relatively small errors ε. Thus, for Examples 1 through 7 the errors are $\varepsilon \approx 1$, 2, 2, 5, 2, 4 and 7, respectively.

According to the Central Limit Theorem and the assumptions already made, a normal distribution of probabilities around the mean path is expected. Thus, a line with a slope $-1/\left(dY/dX\right)$, normal to the mean path at step n, will determine two points at both sides of the mean path at a distance $m\sqrt{n}$ which will embrace the mean trajectory with a region containing a certain number of all possible cases. The Examples 4 (the spiral) and 7 (the number 8) have dotted lines embracing 99.7% of the experiments with a distance $3\sqrt{n}$ at both sides and normal to the mean path.

4 Conclusions

A walker may choose at random the X- or Y-axis when walking on a plane; he may also choose at random a positive or a negative direction along either of these axes. If we assume that the possibilities of these elections are given functions of the step the walker is performing, it is shown that the mean path of many walkers may be predicted theoretically.

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