Inertial random walkers on a surface

Horacio A. Caruso

Departamento de Hidráulica
Facultad de Ingeniería, Universidad Nacional de La Plata, La Plata, Argentina
hacaruso@uolsinectis.com.ar

Sebastián M. Marotta

Departamento de Fisicomatemática
Facultad de Ingeniería, Universidad Nacional de La Plata, La Plata, Argentina
smarotta@gioia.ing.unlp.edu.ar

(*Recebido: 20 de abril de 2002*)

Abstract: *The classical random walker found in the literature has steps along four orthogonal directions, i.e., North, South, East and West, each one with the same possibility to be chosen. We propose a new type of random walker with changing possibilities. They are encouraged to make a new step in the same direction of the previous step.*

Key words: *random walkers, inertial random walkers, chemotaxis, Brownian motion*

Resumo: *As caminhadas aleatórias clássicas descritas na literatura assumem passos em quatro direções ortogonais, i.e., norte, sul, leste e oeste, todas com a mesma probabilidade. Propomos um novo tipo de caminhada aleatória com probabilidades variáveis. Cada novo passo é encorajado a ser dado na mesma direção que a do prévio.*

Palavras-chave: *caminhadas aleatórias, caminhantes aleatórios inerciais, quimiotaxia, movimento browniano*

1 Introduction

Since "The Problem of the Random Walk" stated by Karl Pearson was published in 1905 [1], random walkers have been applied to the study of several phenomena related to physics, mathematics, chemistry, statistics, biology and other disciplines.

Though Brownian motion is the classical problem simulated with random walkers, many modifications of the original simple concept have spread in all areas of scientific research. There have been fruitful results gathered with the mathematical tool of random walkers and big challenges are still waiting to be faced [2-10].

We have studied random walkers that could walk either on a line or on a surface, making steps along two, four or eight directions, with and without the condition of self-avoidance. We have also studied random walkers walking along the branches of a dichotomous tree. The majority of these random walkers were used to study aggregation phenomena. Some of them were even used to simulate algebraic functions [11-16].

Each step the walker performed had no influence whatsoever over the following step. In other words, each step was an independent event. We will herein call them *non-inertial walkers*. We now propose the study of random walkers who decide to make a step according with the decisions they made in the previous step. We call them *inertial walkers*. A preliminary study of inertial walkers moving along a line may be found in [17].

The available literature shows models that take into account some kind of 'memory' in the behavior of random walkers. A pioneer paper by G. I. Taylor [18] about turbulent diffusion studies a model in which a coin toss decides not the direction of the walker's next step but the persistence of motion. There are also theoretical works related to the definition of walkers with persistence, external bias and other forms of memory [19-24]. Most of these works were inspired by the study of the motion of microorganisms toward a source of food, light or heat and the response of organisms to chemical stimuli, i.e., chemotaxis [25-29].

The inertial walkers herein presented are intended in part to mimic enzymatic and genetic checks and mechanisms of balance that are of utmost importance for the growth and survival of many kinds of cells [29].

All random walkers herein studied do not interfere with each other, i.e., they are independent. Walkers walk along virgin terrain when they start a path. They may make steps in four directions only: North, South, East and West; the possibility to select directions North-South or East-West is $1/2$. The length of each step is unity and the maximum number of steps is $n = NS$ for all of them.

2 Inertial walkers

The origin of the walk for each walker is a point

$$P_0(x_0, y_0) \tag{1}$$

The initial possibilities in order to select directions East or West, North or South are

$$K_{0,x} \quad \text{and} \quad K_{0,y} \tag{2}$$

i.e., positive or negative directions along X- and Y-axes, respectively.

Let us assume the walker made n steps and has already decided the direction North-South or East-West, with probability 1/2. In order to decide if the step is positive or negative, we define two variable possibilities

$$K_{n,x} \quad \text{and} \quad K_{n,y} \tag{3}$$

for the X- and for the Y-axes, respectively. A pseudo-random number r is then selected and the rules for step $n+1$ are

$$\begin{array}{llll} \text{if } 0 \leq r \leq K_{n,x} & \text{then} \quad X_{n+1} = X_n + 1 & \text{and} & K_{n+1,x} = K_{n,x} + \varepsilon_x \\ \text{if } K_{n,x} < r \leq 1 & \text{then} \quad X_{n+1} = X_n - 1 & \text{and} & K_{n+1,x} = K_{n,x} - \varepsilon_x \end{array} \tag{4}$$

or

$$\begin{array}{llll} \text{if } 0 \leq r \leq K_{n,y} & \text{then} \quad Y_{n+1} = Y_n + 1 & \text{and} & K_{n+1,y} = K_{n,y} + \varepsilon_y \\ \text{if } K_{n,y} < r \leq 1 & \text{then} \quad Y_{n+1} = Y_n - 1 & \text{and} & K_{n+1,y} = K_{n,y} - \varepsilon_y \end{array} \tag{5}$$

The quantities ε_x and ε_y are small and positive numbers set as initial conditions (for $\varepsilon_x < 0$ and $\varepsilon_y < 0$ for linear inertial walkers see [17]). Notice that with the rules given in equations (4) or (5), the possibilities $K_{n+1,x}$ and $K_{n+1,y}$ change along the sojourn of each inertial random walker.

Let us assume that a random walker makes a step along the X-axis and to the East, i.e., a positive step. Our way to give him the property of a so-called inertia is to 'encourage' him to make another step to the East. Notice that we do not force him to repeat the same kind of step; we simply tell him that he might make another positive step. It will be a matter of chance whether he accepts or not the suggestion. If the step is negative and along the Y-axis, for example, we encourage him to repeat the same kind of step, i.e., we say that, if he feels to do so, he might also make the following step to the South. Due to this peculiar behavior the new type of random walkers we are dealing with could also be called 'memorious' walkers.

It is evident that in the process of increasing or decreasing possibilities in equations (4) and (5), minimum $K_{\min,x}$, $K_{\min,y}$ and maximum $K_{\max,x}$, $K_{\max,y}$, possibilities may eventually be reached. We herein propose that these limits should not be exceeded; thus we set the conditions

$$\text{if} \quad K_{n,x} < K_{\min,x} \quad \text{or} \quad K_{n,x} > K_{\max,x} \quad \text{then} \quad K_{n+1,x} = K_{0,x} \tag{6}$$

and

$$\text{if} \quad K_{n,y} < K_{\min,y} \quad \text{or} \quad K_{n,y} > K_{\max,y} \quad \text{then} \quad K_{n+1,y} = K_{0,y} \tag{7}$$

The possibilities $K_{\min,x}$, $K_{\max,x}$, $K_{\min,y}$ and $K_{\max,y}$ are also initial conditions for all walkers.

Non-inertial walkers may be considered a particular case in which two constant possibilities, $K_{0,x}$ and $K_{0,y}$, are defined and followed throughout their paths.

3 Periods

As we had seen in previous studies of inertial linear walkers [17], we may expect something to occur when walkers meet conditions given in equations (6) and (7), because they denote a step in which walkers change their behavior drastically. These events are herein called 'periods'; they may also be viewed as 'wavelengths.' They are met when steps, of length equal to unity, coincide with coordinates given by

$$T_x = \frac{K_{0,x} - K_{\min,x}}{\varepsilon_x} = \frac{K_{\max,x} - K_{0,x}}{\varepsilon_x} \tag{8}$$

and

$$T_y = \frac{K_{0,y} - K_{\min,y}}{\varepsilon_y} = \frac{K_{\max,y} - K_{0,y}}{\varepsilon_y} \tag{9}$$

or their multiples. The periods are the number of positive steps necessary to increase the possibilities from K_0 to $K_{\max}$ for X and Y directions; the period is also the number of negative steps necessary to decrease the possibility from K_0 to $K_{\min}$ also for X and Y.

We may assume that the inertial random walkers we are dealing with are balls of billiard, elementary particles, electromagnetic fields, bacteria, etc. Equations (4) and (5) could represent a 'tunneling effect'. Equations (6) and (7) are intended to represent a slap in the face or a head-on collision against an obstacle that returns the enthusiastic walker to its initial possibilities. If we think about Brownian motion, ε_x and ε_y could be interpreted as a measure of gas temperature or density.

Periods $T_x = T_y = 0$, that correspond to non-inertial walkers, may be viewed as a case in which $K_{\min,x} = K_{\max,x} = K_{0,x}$ and $K_{\min,y} = K_{\max,y} = K_{0,y}$. Periods $T_x = T_y \to \infty$, with $\varepsilon_x = \varepsilon_y \to 0$, also correspond to non-inertial walkers.

4 Distances reached by walkers

The initial position of all walkers is, from Equation (1), $P_0(x_0, y_0)$; the final site, after NS steps, for a particular walker, is $P_{w,NS}(x_{w,NS}, y_{w,NS})$. An end-to-end mean distance could be defined as

$$\delta = \frac{1}{NW} \sum_{w=1}^{NW} \sqrt{(X_{w,NS} - X_0)^2 + (Y_{w,NS} - Y_0)^2} \tag{10}$$

where NW is the total number of walkers.

5 Simplifications

In all the numerical experiments that follow, we will work with some simplifications to the equations in order to study 'isotropic' random walkers, i.e., walkers with the same properties along both axes X and Y. We will assume that possibilities are $K_{0,x} = K_{0,y} = K_0 = 0.5$, and $K_{\min,x} = K_{\min,y} = K_{\min} = 0$ and $K_{\max,x} = K_{\max,y} = K_{\max} = 1$; increments are $\varepsilon_x = \varepsilon_y = \varepsilon$. In consequence, $T_x = T_y = T$.

6 Two introductory examples

Figure 1 shows $NW = 28$ non-inertial random walkers, each one with $NS = 3000$ steps. The origins of each walk are distributed at random in a square region of sides equal to 500 units of length. Their main characteristic is the typical wiggling of their paths. Due to the aspect of these walks, no care has been taken to give information about their point of origin since it would be irrelevant. It should be stressed that each walk, if stretched, would be $NS = 3000$ units of length. Instead, from equation (10), $\delta \approx 50 \approx \sqrt{NS}$. These two lengths, 3000 and 50, give an idea of how wrapped up or folded their paths are.

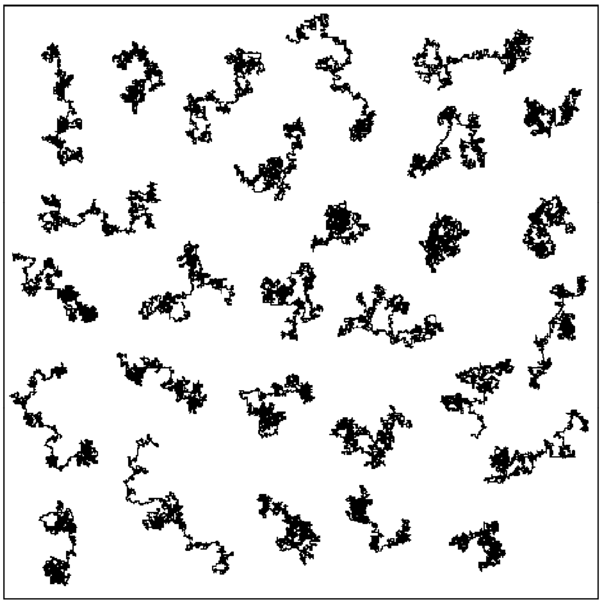

Figure 1. $NW = 28$ non-inertial random walkers with $NS = 3000$ steps. Each one of them starts its path from different points chosen at random on a surface herein limited by a frame of 500 units of length. The end-to-end distance they reach is different for each walker; at the lower line of walkers there is one with a distance of 113 units of length; the walker at the bottom line (extreme right) reaches only 9 units. See Figure 3 for a reliable mean value δ.

Figure 2 has the paths of $NW = 28$ inertial random walkers, also with $NS = 3000$ steps for each one. Their initial conditions are: $K_0 = 1/2$, $K_{\max} = 1$, $K_{\min} = 0$ and $\varepsilon = 0.002$; with these values and Equations (8) and (9), the period becomes $T = 250$. The region where all paths are contained is a square of sides equal to 2000, i.e., four times larger than in Figure 1. The end-to-end mean distance, given by Equation (10), is $\delta \approx 375$. Compare this result for an inertial walker with $\delta \approx 50$ for the non-inertial one; they anticipate an important property of inertial walkers: they reach much longer distances than non-inertial ones. We will return to this property later on.

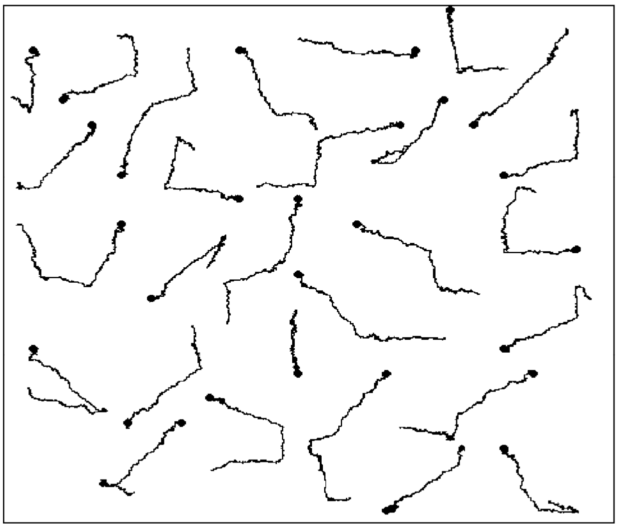

Figure 2. $NW = 28$ inertial walkers with $NS = 3000$ steps start their paths on a region limited by a frame of 2000 units. The increment of probabilities is $\varepsilon = 0.002$. $K_{\max} = 1$, $K_{\min} = 0$ and $K_0 = 0.5$; therefore the period is $T = 250$. Notice how different these trajectories are from those of non-inertial walkers of Figure 1. End-to-end distances vary over wide ranges; see a good average δ in Figure 4.

The paths of these inertial walkers are not so irregular if compared with those of the previous case. Their aspect is entirely different: they show long stretches in what seem to be straight lines; these would be the abovementioned tunneling effect. All paths are zigzagging, i.e., they are far from straight lines, but the scale of the drawing simulates paths with sectors along a line. Some of the samples, not all of them, seem to have an abrupt change of direction; these are the places where the walkers meet conditions of equations (6) and (7), i.e., the slap in the face. We have denoted with a full circle the origin of the path of each inertial walker.

Although we do not intend to make an accurate comparison between the paths followed by our inertial walkers and the trajectories followed by *Escherischia coli*, we may see some similarities. When talking about these bacteria, Berg and Brown [25] say: "The motion appears as an alternating sequence of intervals during which changes in direction are gradual or abrupt – we call these 'runs' and 'twiddles', respectively."

There are all kinds of walks in Figures 1 and 2, and we may draw a good idea of how differently they behave only by inspecting these two figures. However, better information is obtained if the number NW of independent walkers is increased and if the origin of all walks is a single point in the region. This is accomplished in Figures 3 and 4, for non-inertial and inertial walkers, respectively. The former shows the characteristic disorder of classical non-inertial walkers, with a high density of places visited near the origin of all walks and a scarce number of walkers who make long excursions far from the origin. Inertial walkers, on the contrary, show a certain degree of disorder but they seem to prefer visits on some specific places. These sites are at the intersection of vertical and horizontal lines separated by multiples of the period $T = 250$.

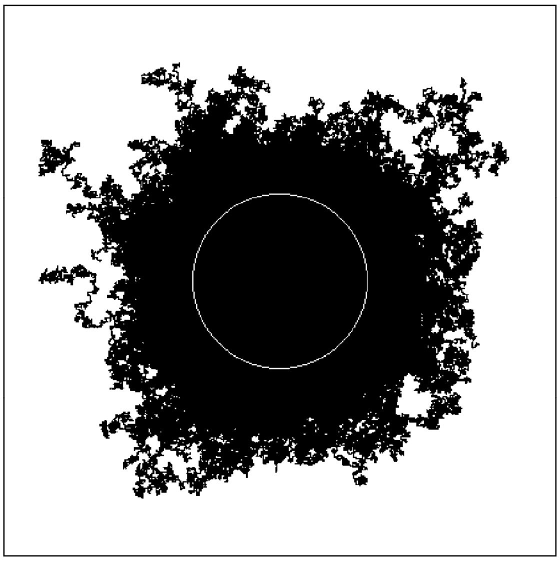

Figure 3. $NW = 1000$ non-inertial walkers with $NS = 6500$ steps. All of them start their paths from the same point, at the center of region limited by a frame of 450 units of length. The mean distance they reach is shown with a white circle of radius $\delta = 71$ units.

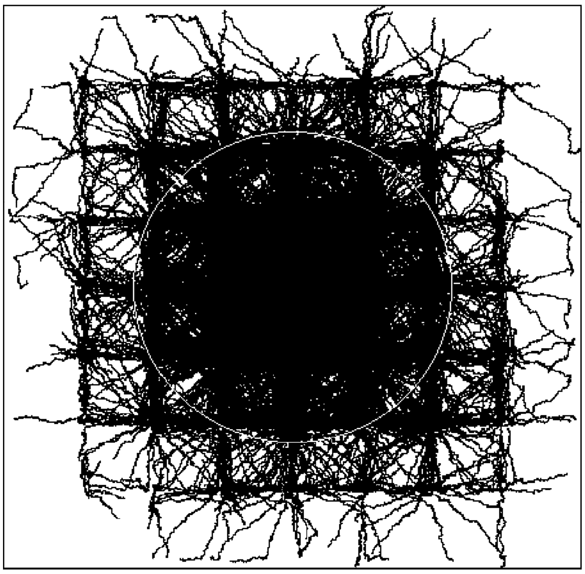

Figure 4. $NW = 1000$ inertial random walkers with $NS = 6500$ steps. All of them start at the center of the region limited by a frame of 2000 units. The mean distance reached by the walkers is shown with a white circle of radius $\delta = 571$ units of length.

7 Frequency distribution of visits

The sites visited by walkers are one unit of length apart from each other, which is also the length of each step. We will study $NW = 10000$ independent walkers, each one with $NS = 7000$ steps; consequently, there will be $NS \times NW = 70 \times 10^6$ visits.

Figure 5 shows the frequency distribution of non-inertial walkers, with the well known sharp peak at the origin and a smooth tendency to zero far from it. This distribution of frequencies is a sum of Gaussian surfaces because we include all the steps of the walkers.

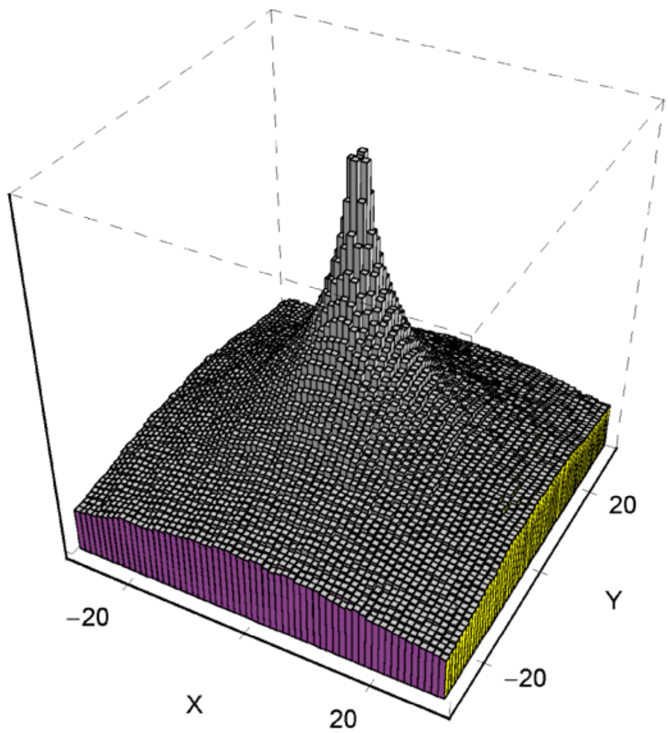

Figure 5. Frequency distribution of visits along X- and Y-axes generated by $NW = 10000$ non-inertial walkers with $NS = 7000$ steps. Notice the characteristic sum of Gaussian surfaces with a sharp peak at the origin. The maximum frequency is 26634 and the mean end-to-end distance is $\delta = 73.7$.

Figure 6 also corresponds to a total of $NW = 10000$ inertial walkers, each with $NS = 7000$ steps; their inertial properties are identified by $K_0 = 0.5$, $K_{\max} = 1$, $K_{\min} = 0$, $\varepsilon = 0.025$ and $T = 20$.

The top of the hills of the landscape show peaks of high frequencies; the surrounding valleys denote sites of lower frequencies of visits. The total number of possible visits is also $NS \times NW = 70 \times 10^6$. The site with a maximum number of visits (12327) is the origin of coordinates; neighboring peaks (to the North, South, East and West) are visited a lower number of times; those to the North-East, North-West, South-East and South-West are visited with still lower number of times. After these second highest peaks and further away from the origin there are many peaks with lower frequencies.

We may view the sites with peaks of frequencies as pits or holes, i.e., places where walkers enjoy to stay more time than in others, a situation suggested by mathematical models in chromatography [4, 9, 10].

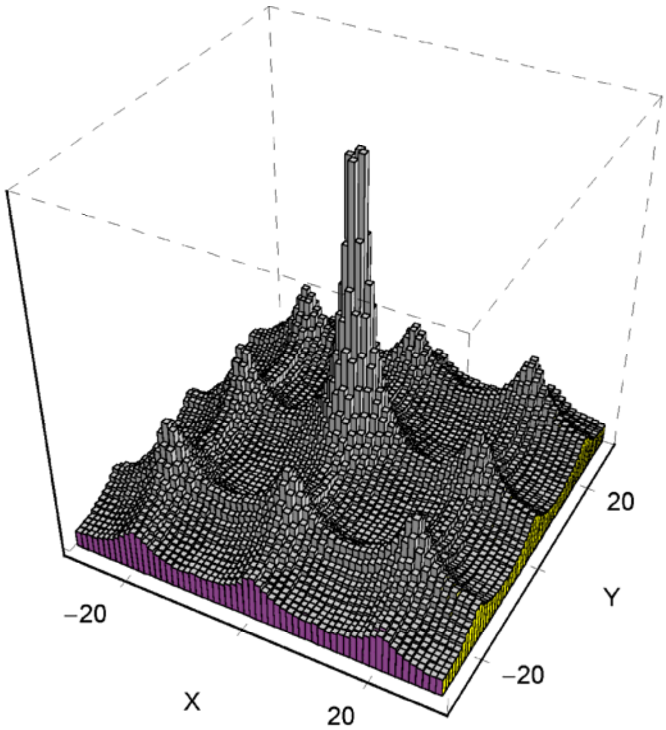

Figure 6. Frequency distribution of visits along X- and Y-axes of $NW = 10000$ inertial walkers each one with $NS = 7000$ steps and $\varepsilon = 0.025$. $K_{\max} = 1, K_{\min} = 0$, and $K_0 = 0.5$; therefore the period is $T = 20$. Notice the sharp peaks separated by multiples of the period in both X- and Y- directions. The maximum frequency is 12327 and the mean end-to-end distance is $\delta = 214.7$.

8 Relation between distances, periods and number of steps

It may be of interest to determine the form of the function

$$\delta = \Phi(T, NS) \tag{11}$$

that relates the distance reached by inertial walkers, the period and the number of steps. It is shown in Figure 7 for $NS = 2500, 2000, 1500, 1000$ and 500.

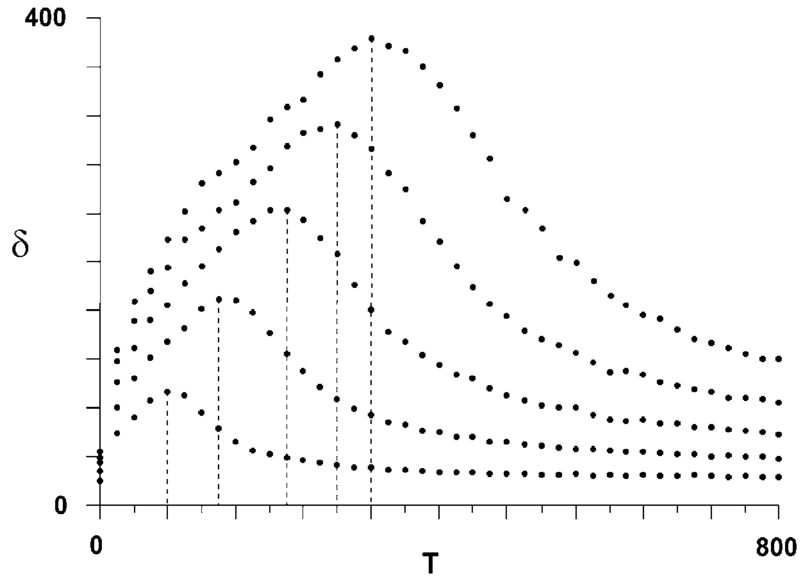

Figure 7. End-to-end mean distance as a function of the period T, for constant values of *NS*; from top to bottom: $NS = 2500, 2000, 1500, 1000$ and 500 steps. Each full circle corresponds to $NW = 1000$ walkers. Dotted lines show the critical periods T_c for which distances reach a maximum, $\delta_{\max}$. $T = 0$ and $T \to \infty$ denote non-inertial walkers which correspond to $\delta_{\min}$. An important observation of these curves is that all distances reached by inertial walkers are greater than those of the non-inertial ones.

It may be seen that, for a constant number of steps, NS, distances are a minimum, $\delta_{\min}$, for $T = 0$ and for $T \to \infty$, i.e., for non-inertial walkers.

For inertial walkers and for the range $0 < T < \infty$ an interesting phenomenon is found: There is a particular period, herein called critical period, T_c, for which walkers reach a maximum distance, $\delta_{\max}$. The critical period is shown with dotted lines in Figure 7.

A relatively large number of numerical experiments allows to express

$$T_c = 0.0851\, NS + 288.22 \tag{12}$$

from Figure 8, and

$$\delta_{\max} = 0.1054\, NS + 314.46 \tag{13}$$

from Figure 9. Both equations are valid for the interval $5000 \leq NS \leq 100000$.

From the two previous equations, $T_c = 0.81\, \delta_{\max} + 28$, or, $\dfrac{T_c}{\delta_{\max}} \approx 0.81$.

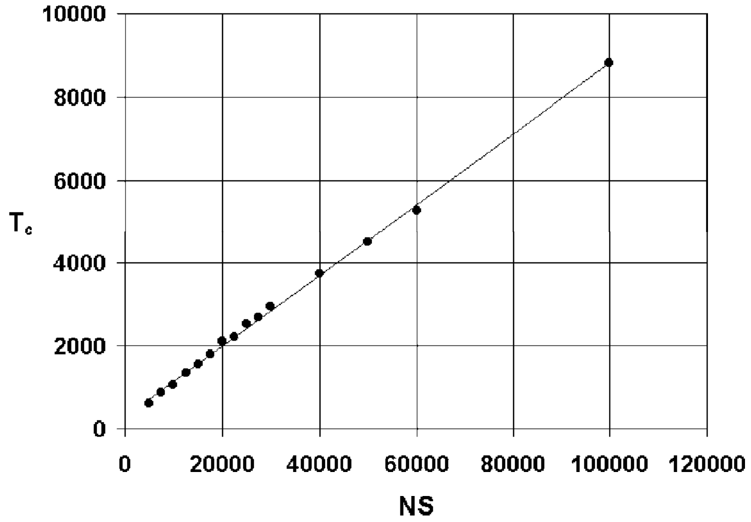

Figure 8. T_c as a function of *NS* in the interval $5000 \leq NS \leq 100000$.
A good fit is $T_c = 0.0851\, NS + 288.22$.

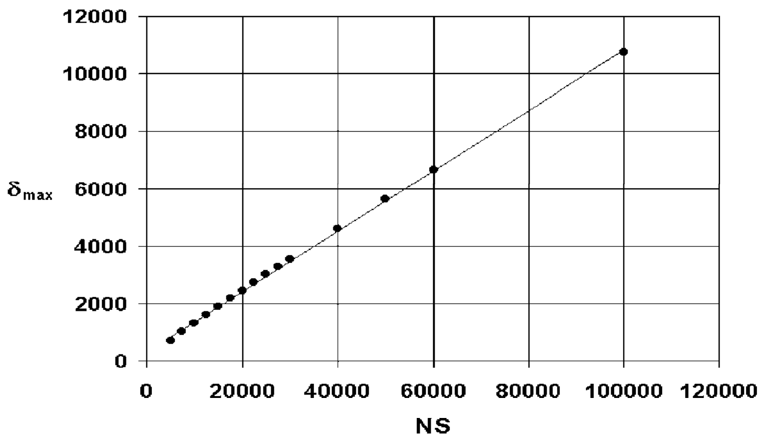

Figure 9. $\delta_{\max}$ as a function of *NS* in the interval $5000 \leq NS \leq 100000$.
The linear function, $\delta_{\max} = 0.1054\, NS + 314.46$, fits in a reasonable manner.

Let us assume that the inertial random walkers we are herein studying are bacteria. Taking into account the time bacteria has survived in our planet we should have respect for their capacity to choose the best way to use their energy when looking for food; their energy could be measured with the number of steps, *NS*. If bacteria need a maximum distance with their available energy we are tempted to assume that these alive creatures might choose an inertial behavior with a critical period.

9 Conclusion

Random walkers who forget the direction of the previous step and have no influence upon the following step may be called non-inertial walkers. On the other hand, random walkers who take into account the direction of the previous step when they are about to make a new step may be called inertial walkers.

It is found that inertial walkers may reach longer distances than non-inertial walkers. According with the way inertia has been given to the walkers in this paper there are sites that are visited with more frequency than others. The distances between peaks of frequency are herein called periods. Numerical experiments show that there is a critical period for which the distance reached by inertial walkers is a maximum.

References

References 14 through 17 are internal publications that can be requested from the Library of the Departamento de Hidráulica de la Facultad de Ingeniería, Universidad Nacional de La Plata, Argentina.

[1] Pearson, K. The problem of the random walk. *Nature 72*,1905, p. 294.

[2] Feynman, R. *Lectures on Physics*. Fondo Educativo Interamericano, 1971.

[3] Haken, H. *Synergetics*. Springer-Verlag, 1983.

[4] Weiss, G. H. Random walks and their applications. *American Scientist* 71, 1983, p. 65-71.

[5] Slade, G. Random walks. *American Scientist* 84, 1996, p. 146-153.

[6] Lavenda, B. H. Brownian motion. *Scientific American* 252, 1985, p. 70-85.

[7] Hayes, B. Prototeins. *American Scientist* 86, 1998, p. 216-221.

[8] Hayes, B. How to avoid yourself. *American Scientist* 86, 1998, p. 314-319.

[9] Giddings, J. C. and Eyring, H. A molecular dynamic theory of chromatography. *J. Phys. Chem* 59, 1956, p. 416-421.

[10] Weiss, G. H. Contributions to the stochastic theory of chromatographic kinetics. *Separation Science* 5, 1970, p. 51-62.

[11] Caruso, H. A. and Marotta, S. M. Self-avoiding random walkers with steps along eight possible directions (octopus walkers). *Revista Ciências Exatas e Naturais*, Ano 2, n. 1, 2000, p. 61-71.

[12] Caruso, H. A. and Núñez, J. Area, perimeter, density and entropy of objects generated by deposition of particles. *Revista Ciências Exatas e Naturais*, Ano 1, n. 2, 2000, p. 9-31.

[13] Caruso, H. A. Simulation of functions Y=f(X) by means of random choices of increments DX and DY (Method of expectancies). *Revista Ciências Exatas e Naturais*, Ano 1, n. 1, 1999, p. 9-22.

[14] Caruso, H. A. and Núñez, J. *Objects generated by oriented random walkers. Some of them may invade the cluster.* May, 1998.

[15] Caruso, H. A. *Objects generated by a dichotomous random walker* (*Chimpanzee walker*) Part I, II and III. 2000-2001.

[16] Caruso, H. A. and Marotta, S. M. *Objects generated by deposition, diffusion and aggregation (DDA) of particles by dichotomous random walkers* Parts I, II and III, 2001.

[17] Caruso, H. A. and Marotta, S. M. *Sites visited by a linear random walker*, PartI through VI and Part A, 2000-2001.

[18] Taylor, G. I. Diffusion by continuous movements. *Proc. Lond. Math. Soc.* 20, 1920, p. 196-212.

[19] Nossal R. *Mathematical theories of topotaxis, in biological growth and spread.* Mathematical Theories and Applications, ed. W. Jager, H. Röst, and P. Tautu, Springer-Verlag: 1980, p. 410-439.

[20] Patlak, C. S. Random walk with persistence and external bias. *Bull. Math. Biophys*, 15, 1953, p. 311-338.

[21] Nossal, R. J. and Weiss, G. H. A generalized pearson random walk allowing for bias. *Journal of Statistical Physics* 10, 1974, p. 245-253.

[22] Hara, H. Generalization of the random-walk process. *Physical Review B* 20, 1979, p. 4062-4068.

[23] Gillis, J. Correlated random walk. *Proc. Cambridge Philosophical Soc.* 51, 1955, p. 639-651.

[24] Davis, B. Reinforced and perturbed random walks. *Bolyai Society Mathematical Studies* 9, 1999, p. 113-126.

[25] Berg, H. C. and Brown, D. A. Chemotaxis in escherischia coli analysed by three-dimensional tracking. *Nature* 239, 1972, p. 500-504.

[26] Macnab, R. M. and Koshland JR., D. E. The gradient-sensing mechanism in bacterial chemotaxis. *Proc. Nat. Acad. Sci. USA* 69, 1972, p. 2509-2512.

[27] Nossal, R. and Zigmond, S. H. Chemotropism indices for polymorphonuclear eukocytes. *Biophysical Journal* 16, 1976, p. 1171-1182.

[28] Gail, M. H. and Boone, C. W. The locomotion of mouse fibroblasts in tissue culture. *Biophysical Journal* 10, 1970, p. 980-993.

[29] Kessin, R. H. and Van Lookeren Campagne, M. M. The development of a social amoeba. *American Scientist* 80, 1992, p. 556-565.