The spiked harmonic oscillator revisited
New results

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Abstract: *An economical method discovered by Znojil to find the limiting value of a particular function, of interest in problems like the spiked harmonic oscillator, is applied and generalized. The result thus obtained is then applied to find closed form expressions for the sums of some infinite series involving gamma functions.*

Key words: *spiked harmonic oscillator, gamma function*

Resumo: *Aplica-se e generaliza-se um método econômico descoberto por Znojil para encontrar o valor limite de uma função particular de interesse em problemas como o do oscilador harmônico agudo. O resultado assim obtido é, então, aplicado para encontrar expressões em forma fechada para somas de algumas séries infinitas envolvendo funções gama.*

Palavras-chave: *oscilador harmônico agudo, funções gama*

1 Introduction

Since the pioneering papers by Ezawa *et al.* [1], and Detwiler and Klauder [2] on supersingular potentials, an extensive literature has been developed on the subject. The supersingular potential behaves abnormally in enough ways to exclude it from being classified as a regular potential. More specifically, by a supersingular potential is meant the particular potential which has the property that every matrix element of the perturbation potential with respect to the unperturbed eigenstates is infinite or does not exist [2].

The spiked harmonic oscillator system Hamiltonian reads as follows

$$H = -\frac{d^2}{dr^2} + r^2 + \frac{\lambda}{r^\alpha} \tag{1}$$

where λ is a definite parameter which measures the strength of the singular potential, and α is a positive constant defining the degree of the singularity.

The Hamiltonian (1) renders the phenomenon of supersingularity when the exponent α is such that $\alpha \geq 5/2$. For $\alpha < 5/2$, Eq. (1) defines the nonsingular spiked harmonic oscillator (NSHO) problem which has been a subject of an extensive study in the past few years [3-5]. In Ref. [3], Aguilera-Navarro and Guardiola introduced a function F defined as

$$F = \sum_{n=0}^{\infty} \frac{(-\gamma/2)_n^2}{4(n+1)(3/2)_n n!} = \frac{1}{2\gamma^2}\left[{}_2F_1\left(-\gamma,-\gamma;\frac{1}{2};1\right) - 1 - 2\gamma^2\right] \tag{2}$$

where $\gamma = \alpha/2 - 1$ and $(z)_n$ is the Pochhammer symbol.

Clearly, F is not defined for $\alpha > 7/2$, i.e. $\gamma > 3/4$, and requires careful treatment when $\alpha = 2$, since, in this case, $\gamma = 0$. Aguilera-Navarro and Guardiola [3] evaluated F for $\alpha = 1/2$, $\alpha = 1$, and $\alpha = 3/2$ and the results were employed in the corresponding determination of the analytic approximations for the ground-state energy of the NSHO for the same values of α. The evaluation of the function F for the limiting case $\alpha = 2$ was treated separately shortly thereafter [6]. A more simple and singularly elegant method for doing the latter evaluation of the function F was recently discovered by Znojil [7].

In this work we employ Znojil's method [7] to evaluate the limit of a function closely related to F. Our generalization turns out to be quite useful for the evaluation in closed form of infinite series whose terms involve ratios and products of gamma functions.

2 Procedure

We purport to show that the following relation holds

$$L(c,x) = \lim_{\gamma \to 0}\left[{}_2F_1(-\gamma,\gamma;c;x) - 1\right]\gamma^{-2} = -\Gamma(c)\sum_{n=1}^{\infty}\frac{\Gamma(n)x^n}{n\,\Gamma(n+c)} \tag{3}$$

In order to find $L(c,x)$, we set $a = b = \epsilon$ in the usual definition of the Gauss hypergeometric function [8]

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\,\Gamma(n+b)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(n+c)} \frac{x^n}{n!} \tag{4} $$

to get

$$ L(c,x) = \lim_{\epsilon \to 0} \left[\sum_{n=0}^{\infty} \frac{\Gamma^2(n+\epsilon)\,\Gamma(c)}{\Gamma^2(\epsilon)\,\Gamma(n+c)} \frac{x^n}{n!} - 1 \right] \epsilon^{-2} \tag{5} $$

The term corresponding to $n = 0$ in the infinite series cancels with the negative one within the square brackets yielding

$$ L(c,x) = \lim_{\epsilon \to 0} \left[\sum_{n=1}^{\infty} \frac{\Gamma^2(n+\epsilon)\,\Gamma(c)}{\epsilon^2\,\Gamma^2(\epsilon)\,\Gamma(n+c)} \frac{x^n}{n!} \right] \tag{6} $$

Since

$$ \lim_{\epsilon \to 0} \Gamma(n+\epsilon) = \Gamma(n) \text{ and } \lim_{\epsilon \to 0} \epsilon\,\Gamma(\epsilon) = \lim_{\epsilon \to 0} \Gamma(\epsilon+1) = 1 \tag{7} $$

it follows that the sought for limit is given by

$$ L(c,x) = \sum_{n=1}^{\infty} \frac{\Gamma^2(n)\,\Gamma(c)}{\Gamma(n+c)} \frac{x^n}{n!} \tag{8} $$

Finally, the relation $\Gamma(n) = (n-1)!$ leads to the relation (3) above.

We are now ready to evaluate in *closed form* the sum in (3), when the argument x equals *1*. In other words, we want to find a closed expression for the following infinite sum

$$ S(c) = \sum_{n=1}^{\infty} \frac{\Gamma(n)\,\Gamma(c)}{n\,\Gamma(n+c)} = \lim_{\epsilon \to 0} \left[{}_2F_1(\epsilon,\epsilon;c;1) - 1 \right] \epsilon^{-2} \tag{9} $$

The special case $S(1)$ follows immediately, since $\Gamma(n+1) = n\,\Gamma(n)$, namely,

$$ S(1) = \lim_{\epsilon \to 0} \left[{}_2F_1(\epsilon,\epsilon;1;1) - 1 \right] \epsilon^{-2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \tag{10} $$

where $\zeta(z)$ is the Riemann zeta function.

The evaluation in closed form of Eq. (9) for other values of c could be accomplished using a simple but somewhat long method employing the power series of $\ln \Gamma(1+z)$. The most straightforward route known, however, suggested to us by a paper by Znojil [7], will be employed in this work.

Setting $a = b = \epsilon$ in the well-known expression [9]

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{11} $$

and substituting the result in Eq. (9) yields

$$S(c) = \lim_{\epsilon \to 0} \left[\frac{\Gamma(c)\,\Gamma(c-2\epsilon)}{\Gamma^2(c-\epsilon)} - 1 \right] \epsilon^{-2} \tag{12}$$

The Maclaurin's expansion of the gamma function reads as [7]

$$\Gamma(c+x) = \Gamma(c)\left\{ 1 + x\psi(c) + \frac{1}{2}x^2 \left[\psi^2(c) + \psi^{(1)}(c) \right] + \cdots \right\} \tag{13}$$

where $\psi(y)$ and $\psi^{(n)}(y)$, $n \geq 1$, are the digamma and the polygamma functions [10], respectively.

Employing the expansion (13) twice in Eq. (12) and retaining within the square brackets only powers of ϵ up to ϵ^2, the following main result is readily found

$$S(c) = \sum_{n=1}^{\infty} \frac{\Gamma(c)\,\Gamma(n)}{n\,\Gamma(n+c)} = \lim_{\epsilon \to 0} \left[{}_2F_1(\epsilon, \epsilon; c; 1) - 1 \right] \epsilon^{-2} = \psi^{(1)}(c) \tag{14}$$

where $\psi^{(1)}(y)$ is the trigamma function [11] defined by

$$\psi^{(1)}(y) = \sum_{n=0}^{\infty} \frac{1}{(n+y)^2} \tag{15}$$

The infinite series in Eq. (15), known in the mathematical literature as Hurwitz's generalization of the zeta function, has been extensively studied [8, 9], [12-14].

It is worth applying the result Eq. (14) to get the following special cases

$$S(m) = \sum_{n=1}^{\infty} \frac{\Gamma(m)\,\Gamma(n)}{n\,\Gamma(n+m)} = \frac{\pi^2}{6} - \sum_{k=1}^{m-1} \frac{1}{k^2}, \quad m = 1, 2, 3, \ldots \tag{16}$$

and

$$S(m+1/2) = \sum_{n=1}^{\infty} \frac{\Gamma(m+1/2)\,\Gamma(n)}{n\,\Gamma(n+m+1/2)} = \frac{\pi^2}{2} - 4\sum_{k=1}^{m} \frac{1}{(2k-1)^2}, \quad m = 0, 1, 2, \ldots \tag{17}$$

3 Concluding remarks

Our main resul, Eq. (14), is not only compact but also bears a striking resemblance to the function F of Aguilera-Navarro and Guardiola [3]. When $c = 1$, our Eq. (14) becomes Eq. (10) which is just an alternative way of writing a particular case of Euler's dilogarithm function [13]. Moreover, Eq. (3) with $c = 1$ is the dilogarithm function of argument x. We have decided to include the derivation of Eq. (3) with $c = 1$ not only for the sake of completeness, but also because Mitchell [12], who first published this relation, did not show explicitly how to prove it.

It is perhaps worth recording that the dilogarithm function arises in topics such as the analysis of a distorted modulated electric signal [15], as well as in a variety of other problems of physical and mathematical interest. Among these we mention the problem of the relation between amplitude and phase in an electrical circuit [16], the scattering of light by light [17], heat transfer on a circuit cylinder [18], and certain problems involving the Fermi function [19].

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