Sequences of complex numbers resembling the Fibonacci series

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Abstract: *Each term of the classical Fibonacci sequence of numbers is the sum of the two previous terms of the sequence. If instead of the sum the third term is an addition or a subtraction of the two previous terms, one of them multiplied by a constant, new and rich sequences are obtained. Some of the properties of these sequences are herein studied by means of numerical procedures, incorporating the condition that each term, including the constant, are complex numbers.*

Key words: *Fibonacci numbers, golden ratio, vibonacci numbers, strange attractors, IFS, fractals, Stern-Brocot tree*

Resumo: *Cada termo da seqüência clássica dos números de Fibonacci é a soma dos dois prévios termos da seqüência. Se, em vez da soma, o terceiro termo é uma adição ou uma subtração dos dois prévios termos, uma delas multiplicada por uma constante, novas e ricas seqüências são obtidas. Algumas das propriedades dessas seqüências são então estudadas por meio de procedimentos numéricos, incorporando-se a condição de que cada termo, incluindo a constante, são números complexos.*

Palavras-chave: *Números de Fibonacci, razão áurea, números de vibonacci, atratores estranhos, IFS, fractais, árvore de Stern-Brocot*

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1 Introduction

The ratio of two consecutive terms in a series of numbers such as $\ldots, f(n-2)$, $f(n-1), f(n), \ldots$, provided $f(n) = f(n-2) + f(n-1)$ leads to the famous and ancient golden number $\phi = \lim_{n \to \infty} [f(n)/f(n-1)] = (1+\sqrt{5})/2 = 1.618\ldots$. A similar series of real numbers such as $\ldots, v(n-2), v(n-1), v(n), \ldots$, with the condition that $v(n) = v(n-2) \pm \beta v(n-1)$, may be defined, where β is a real constant and a $\pm$ sign is chosen at random with equal probabilities. The growth rate of this series, defined by $C = |v(n)|^{1/n}$, has been thoroughly studied by Divakar Viswanath recently [1]. An interesting and instructive introduction to this subject, with many variations, can also be found in Hayes [2].

We herein propose the study of the behavior of a series of complex terms, including β, by means of a ratio of terms similar to that of Fibonacci.

2 Definitions

Let us define a complex number $v(n)$

$$v(n) = v_{\text{real}}(n) + i\, v_{\text{imag}}(n) \tag{1}$$

where $v_{\text{real}}(n)$ and $v_{\text{imag}}(n)$ are its real and imaginary parts, respectively, and $i = \sqrt{-1}$. Let us also define a series of complex numbers,

$$\ldots, v(n-2), v(n-1), v(n), v(n+1), \ldots, v(n_f) \tag{2}$$

where, by definition,

$$v(n+1) = v(n-1) \pm \beta v(n) \tag{3}$$

The constant

$$\beta = \beta_{\text{real}} + i\beta_{\text{imag}} = r_\beta\, e^{i\theta_\beta} \tag{4}$$

is also complex. The symbol $\pm$ means that the second term at the right of equation (3) is added or subtracted, at random and with probability 1/2.

If we divide equation (3) by $v(n)$, and if we define complex ratios with,

$$CV(n) = \frac{v(n)}{v(n-1)} = CV_{\text{real}}(n) + CV_{\text{imag}}(n) \tag{5}$$

then, equation (3) can be regarded as the Iterated Function System (IFS),

$$CV(n+1) = \frac{1}{CV(n)} \pm \beta \tag{6}$$

By plotting $CV(n)$ in the complex plane, as the number of iterations increase, a wide variety of attractors may be obtained for different values of β.

It should be noticed that, through the use of the ratio $CV(n)$, our study becomes independent of the magnitude of each term $v(n)$ which are, in general, large numbers; we thus avoid the special algebra required for big numbers.

The first 500 terms of the series are ignored in order to avoid a possible transient state due to the election of the two initial terms (initiators). Unless otherwise stated, all our numerical results are performed with $n_f = 10□000□000$ terms $CV(n)$ of the series.

In the process of generating the attractor of equation (6), there might be a case in which a term $CV(n)$ may turn out to be zero. In these rare cases, the series is aborted and re-started again with a different series of random choices for the selection of plus or minus signs.

3 Objectives of this study

The stages of the present study of equation (6), resulting from the election of different complex constants for equation (4), starts with the more complete sequence of numbers and ends with the simplest one. The stages, together with the proposed names of the resulting sequences are:

1) □ with $r_□ = 1$ and 0 □ 2□: Vibonacci Sequences of Complex Numbers;

2) □ with $r_□ = 1$ and $_□ = 0$: Vibonacci Sequences of Real Numbers;

3) +□ with 0 □ 2□: Fibonacci Sequences of Complex Numbers, either with $r_□ = 1$ or $r_□ \neq 1$;

4) +□ $r_□ = 1$ with $_□ = 0$: The classical Fibonacci Sequences of Real Numbers.

Due to the vibrating nature of the terms $v(n)$ from negative to positive values, and vice versa, Hayes [2] proposed the name “Vibonacci” for these series.

Each value of □ produces a different type of attractor in the Vibonacci series; for reasons of brevity, only two of them will be herein studied. Furthermore, the probability of selection of plus or minus sign for each term of the series will be kept at 1/2, as it was previously mentioned; different probabilities may be considered of no import to this study.

4 Analysis of results for the vibonacci sequences of complex numbers

The first attractor, herein shown as figure 1(a), corresponds to the constant

$$□ = e^{i□□4} \tag{7}$$

The attractor is limited by two straight and parallel lines inclined with an angle of □□4 with respect to the real axis and two units apart from each other.

Figure 1(b) is the same attractor but with an artifact: those points of the iteration which fall inside a circle of radius equal to unity remain in their places; those points lying outside the circle of unit radius are brought to the interior of the circle by means of the transformation of the circle, $w = 1□z$. The attractor has a certain similitude with the Triangle of Sierpinski, with empty circular zones instead of empty triangles.

Figure 2(a) is the same IFS of equation (6) but with the constant

$$\square = e^{i\square\square 6} \tag{8}$$

The same procedure of bringing the points outside the circle of unit radius to the interior of the circle by means of the transformation of the circle yields figure 2(b).

Figure 1

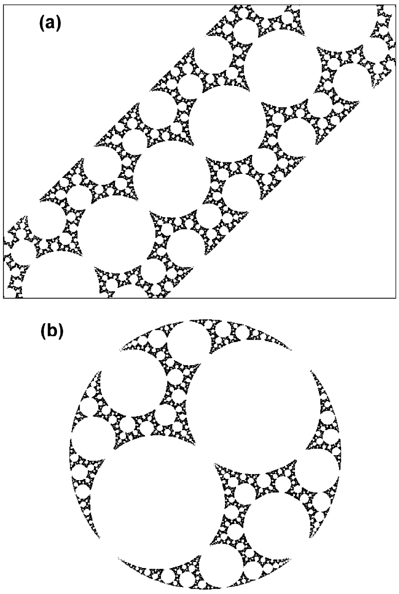

Figure 2

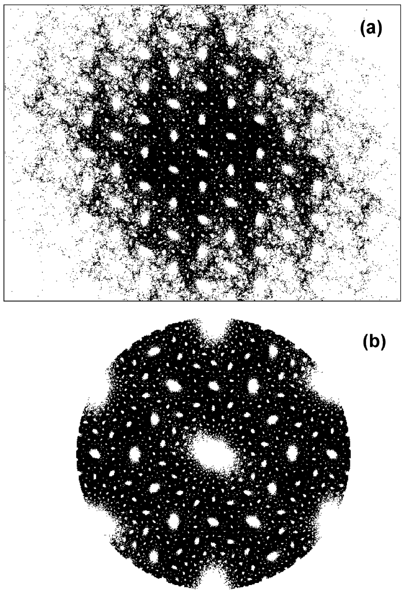

Figure 1. (a) Attractor resulting from the IFS of equation (6) with $\square = e^{i}\square\square 4$; the width of the frame is 5 units. The real part of the complex $CV(n) = \square(n)\square\square(n \square 1) = CV_{\text{real}}(n) + i\,CV_{\text{imag}}(n)$ is plotted along the horizontal axis, and the imaginary part along the vertical axis. (b) the same attractor but those points outside the circle of unit radius are brought to the interior of the circle by means of the well known transformation of the circle ($w = 1\square z$).

Figure 2. Same as in figure 1 but with $\square = \square\square 6$; the frame is 12 units wide.

Figures 1 and 2 suggest the self-similarity of the fractal structure of the attractor. This point will be discussed later.

The polar form of equation (5) may be expressed as

$$CV(n) = r_{CV}(n)e^{i\,\square_{CV}(n)} \tag{9}$$

where the modulus is

$$r_{CV}(n) = \sqrt{CV_{\text{real}}(n)^2 + CV_{\text{imag}}(n)^2} \tag{10}$$

and the argument is

$$\square_{CV}(n) = \tan^{\square 1} \frac{CV_{\text{imag}}(n)}{CV_{\text{real}}(n)} \tag{11}$$

For an IFS of n_f points, the frequency distributions of modulus and arguments can be studied; they are represented in figures 3(a) and 3(b), respectively, for the constant of equation (7). The main characteristic of these frequency distributions is the existence of valleys in which there is a very small probability (or none at all) to find either modulus or arguments. Between these valleys there are peaks of probabilities. The ranges of variation of modulus are $0.107 \le r_{CV}(n) \le 10.328$, and $0 \le \theta_{CV}(n) \le 2\pi$ for arguments, for this particular experiment. Their mean values are $r_{CV\,\mathrm{mean}} = 1.315$ and $\theta_{CV\,\mathrm{mean}} = 199.296^0$. The modulus of maximum frequency is approximately equal to unity, but there are many other modulus with relatively high probabilities. For the particular case of the constant given by equation (7), some of the valleys of the arguments may be found at $\theta_{CV} = m\pi/4$ for $m = 1, 3, 5$ and 7, and some of the highest probabilities are at $\theta_{CV} = m\pi/8$ for $m = 1, 3, 5, 7, \ldots, 15$.

Frequency distributions of modulus and arguments for the pierced attractor of figure 2 are given in figures 4(a) and 4(b), in which case $r_{CV\,\mathrm{mean}} = 1.369$ and the modulus of maximum frequency is of the order of 0.689.

Figure 3

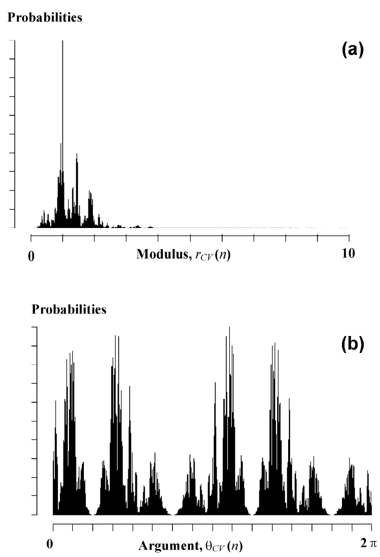

Figure 4

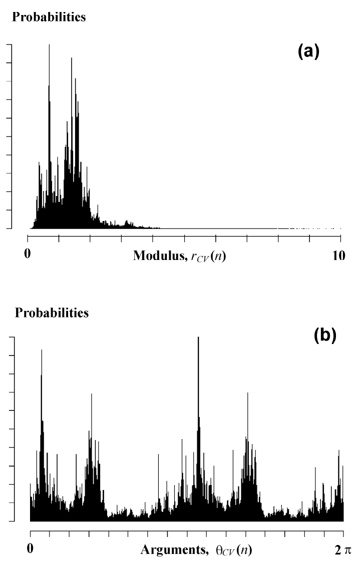

Figure 3. Probability distributions of the complex constant $CV(n) = r_{CV}(n)\, e^{i\,\theta_{CV}(n)}$ for the attractor of figure 1. (a) for modulus in the range $0 \le r_{CV}(n) \le 10$; the maximum probability detected for this particular numerical experiment is 0.0169 and modulus were detected in the range $0.107 \le r_{CV}(n) \le 10.328$. (b) probabilities for arguments between $0 \le \theta_{CV}(n) \le 2\pi$; the maximum probability is 0.0016.

Figure 4. Same as in figure 3 but for the attractor of figure 2. The maximum probability shown in (a) is 0.0050 and 0.0021 in (b).

5 The "history" of each term of the sequences

When iterations of the Vibonacci Series of Complex Numbers progress, the attractor grows as points are laid down in the plane. It is not easy to discover any rhythm in the growth since they may make small or large jumps in the plane, just like any IFS. In spite of this apparent disordered activity, we may find a way to see the general behavior of the growth by studying the modulus at iteration $n+1$ as a function of the modulus at the previous iteration, i.e. the study of the function $r_{CV}(n+1) = f_r(r_{CV}(n))$; the same may be done with arguments with the study of the function $\theta_{CV}(n+1) = f_\theta(\theta_{CV}(n))$. Figures 5(a) and 5(b) are the attractors describing both functions for the constant of equation (7). Figures 6(a) and 6(b) belong to the constant of equation (8).

Figure 5

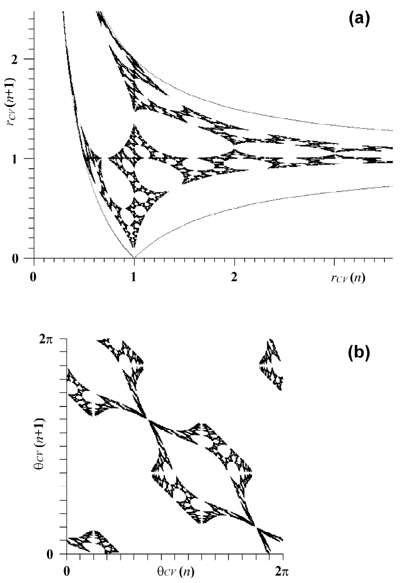

Figure 6

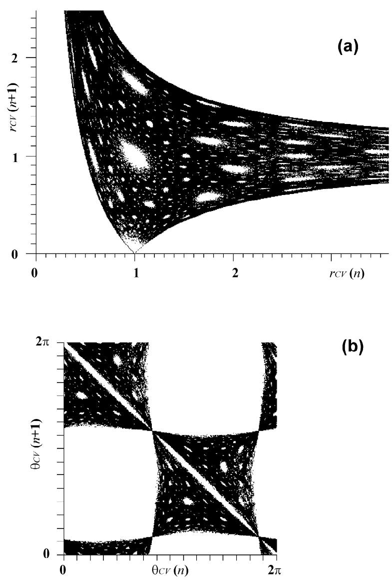

Figure 5. The "history" of the attractor of figure 1 with the function $r_{CV}(n+1) = f_r(r_{CV}(n))$ in (a) and the function $\theta_{CV}(n+1) = f_\theta(\theta_{CV}(n))$ in (b). The boundaries of (a) are the inequalities of equations (12) and (13).

Figure 6. Same as in figure 5 but for the attractor of figure 2.

From figures 5(a) and 6(a) and equation (6), it may be clearly seen that, for a particular $r_{CV}(n)$, the following modulus, $r_{CV}(n+1)$, is bounded by three hyperbolae. The region where $r_{CV}(n+1)$ may be found is limited by the following inequalities:
for $r_{CV}(n) \geq 1$:

$$1 - \frac{1}{r_{CV}(n)} \leq r_{CV}(n+1) \leq 1 + \frac{1}{r_{CV}(n)} \tag{12}$$

and for $r_{CV}(n) \geq 1$,

$$-1 + \frac{1}{r_{CV}(n)} \leq r_{CV}(n+1) \leq 1 + \frac{1}{r_{CV}(n)} \tag{13}$$

It should also be noticed that the empty spaces of figures 5 and 6 denote regions in which a modulus $r_{CV}(n)$ is never followed by another modulus $r_{CV}(n+1)$. Figures 5(a) and 6(a) look like distorted views of figures 1 and 2, respectively.

6 Results for the vibonacci sequences of real numbers

If all the terms of equation (3) are real, and if the constant of equation (4) is also real, with $\alpha = 1$, the result will be the Vibonacci Series of Real Numbers. Thus, it seems licit to replace $CV_{\text{real}}(n)$ by $CV(n)$ in equation (5). Furthermore, since the attractor is symmetrical with respect to $CV(n) = 0$, it is more convenient to represent the absolute value of the ratio of two consecutive terms of the sequence, i.e. $|CV(n)|$. Also, in order to make the attractor more visible, a vertical axis with the number n of the iteration has been added. Figure 7(a) and 7(b) show the attractor for this particular sequence of numbers, for two different ranges of $|CV(n)|$; this experiment yielded constants in the range $-21.2 \leq CV(n) \leq 21.5$.

All points of the attractor lie along the real axis, i.e. the attractors of figure 1 or 2 collapse into a line. The arguments are either zero or π, but the probability distribution of modulus exhibit the same characteristics given in figure 3(a) and 4(a), with their peaks and valleys. Numerical experiments show that $|CV(n)|_{\text{mean}} = 1.618344$. The maximum frequency has approximately the same value, namely, 1.618500. Both of these constants are approximately equal to the golden ratio. It seems that $|CV(n)|_{\text{mean}} \to \phi$, but it does so very slowly.

The inequalities of equations (12) and (13) are transformed into the equalities: for $r_{CV}(n) \leq 1$,

$$r_{CV}(n+1) = 1 + \frac{1}{r_{CV}(n)} \quad \text{or} \quad r_{CV}(n+1) = 1 - \frac{1}{r_{CV}(n)} \tag{14}$$

and for $r_{CV}(n) \geq 1$,

$$r_{CV}(n+1) = 1 + \frac{1}{r_{CV}(n)} \quad \text{or} \quad r_{CV}(n+1) = -1 + \frac{1}{r_{CV}(n)} \tag{15}$$

with $r_{CV}(n) \equiv |CV(n)|$ and $r_{CV}(n+1) \equiv |CV(n+1)|$. It may be seen that equations (14) and (15) are the boundaries of figures 5(a) and 6(a).

In order to see the condition of self-similarity in the probability distribution given in figures 7(a) and 7(b), drawn with a linear scale, a new scale is adopted using the Stern-Brocot tree [3]. The results are given in figure 8; it may be observed that the probability distribution for the complete interval between 0/1 and 1/0 given in (a) is reproduced in (b) for the interval 1/1 and 2/1. Also, for the interval 8/5 and 13/8 given in (c), the probability distributions of $|CV(n)|$ are the same. Note that the peak of maximum frequency stands out at ϕ. This self-similarity has already been described by Viswanath [1,2] coming from a different starting point.

Figure 7

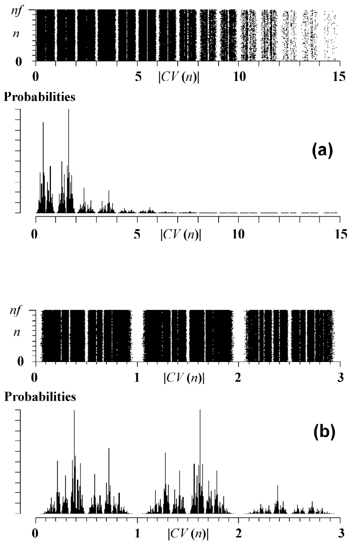

Figure 8

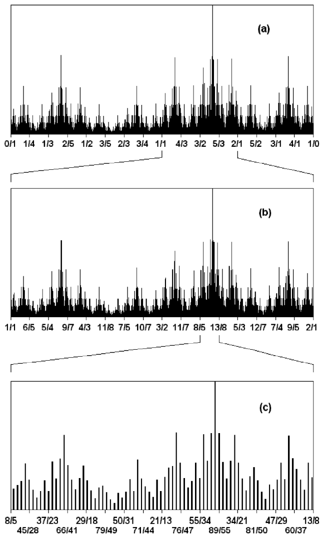

Figure 7. (a) above: attractor of the constant $|CV(n)|$ of a Vibonacci Series of Real Numbers. The vertical axis is the order n of the iteration of the IFS; it is somewhat artificial, but is used in order to show the structure of the attractor with regions devoid of points. Figure 7(a) below: probability distribution of the constant $|CV(n)|$ for the range $0 \leq |CV(n)| \leq 15$; the maximum probability herein detected is 0.0212. Figure 7(b) is a closer view of figure 7(a), for the range $|CV(n)| \leq 3$; the maximum probability is 0.0095. These figures, with a general view of the attractor and a detail, are intended to show a preview of its self-similarity, rigorously demonstrated with the Stern-Brocot tree.

Figure 8. Stern-Brocot tree in order to see self-similarity in the probability distribution of constants $|CV(n)|$; (a) for the complete range $0/1 \leq |CV(n)| \leq 1/0$; (b) and (c) are closer views. The maximum probability is 0.0026.The golden number ϕ is the most frequent.

7 Results for the Fibonacci sequences of complex numbers

We will herein study a complex number similar to the one given in equation (1):

$$f(n) = f_{\text{real}}(n) + i f_{\text{imag}}(n) \tag{16}$$

with a series of complex terms

$$\ldots, f(n-2), f(n-1), f(n), f(n+1), \ldots, f(n_f) \tag{17}$$

obeying the definition

$$f(n+1) = f(n-1) + \beta f(n) \tag{18}$$

The complex constant β is given by equation (4). Note that the terms of the series in equation (3) were added or subtracted, and the decision was taken by the flipping of a coin; in this case they are always added, a fact which transforms the problem into a deterministic one. We may define complex ratios with

$$CF = \lim_{n\to\infty} \frac{f(n)}{f(n-1)} = CF_{\mathrm{real}} + i\, CF_{\mathrm{imag}} \tag{19}$$

Equations (5) and (19) are equivalent, i.e., they have the same meaning. However, in the IFS given by the former, we may have as many values of $CV(n)$ as n_f terms are calculated, while for the latter there is only one constant *CF*. This is true provided n_f reaches a certain value, and that is why we have used *CF* instead of *CF(n)*. This condition will be discussed in the following paragraph.

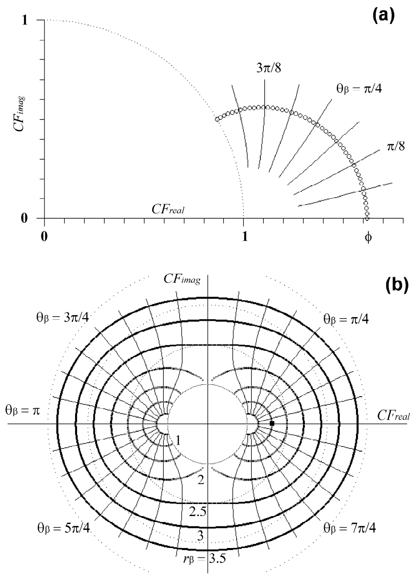

Figure 9. The complex constant $CF = \lim_{n\to\infty} f(n)/f(n-1) = CF_{\mathrm{real}} + i\, CF_{\mathrm{imag}}$ in equation (19) for a Fibonacci Series of complex numbers; *(a)* empty circles denote CF for $r_\beta = 1$ and different arguments θ_β in the complex constant $\beta = r_\beta e^{i\theta_\beta}$; full lines represent CF in which θ_β is constant and r_β changes; a dotted line is a circumference of unit radius. *(b)* the complete description of *CF* for different modulus and arguments of β. A full square indicates the "proportio divina", the "sectio aurea", ϕ.

In figure 9(a) we have represented the values of *CF* of equation (19) with empty circles, i.e. values of *CF* for different θ_β and $r_\beta = 1$. Each full line denote equation (19) for different values of r_β and contant angles θ_β. The dotted line is a circumference of radius equal to unity used as reference.

Let us consider the case in which $\square = e^{i\,45^0}$ in equation (18), with hardly n_f □ 25 terms of the series, the complex of equation (19) converges to the constant $CF = 1\square 443\, e^{i\,19\square 334^0}$, see gure 9(a). For □ = 80^0 the constant is $CF = 1\square 105\, e^{i\,29\square 339^0}$, but it requires n_f □ 80 terms. With □ = 89^0, $CF = 1\square 010\, e^{i\,29\square 993^0}$ provided n_f □ 800. For □ = $89\square 9999^0$□an approximate value of the constant is $CF = e^{i\,30^0}$, but it is reached only after n_f □ 7,000,000 terms. Therefore, we may conclude that, for $\square = e^{i\,90^0}$, the constant $CF = e^{i\,30^0}$ is obtained after an in nite number of terms, i.e. the series is always in a transient state.

The complete landscape of complex constants *CF*, for di□erent modulus and arguments of $\square = r_\square e^{i\,\square\square}$, is represented in gure 9(b). It should be stressed that a single point in gure 9(b) is transformed in a strange attractor in gures 1 and 2.

8 Fibonacci sequences of real numbers

All real constants *CF* are placed along the real axis of gure 9*(b)*, obtained with di□erent modulus and arguments equal to zero in equation (4). Negative constants *CF* come from $\square = r_\square e^{i\square} = \square r_\square$.

One particular constant, coming from a modulus equal to unity and an argument equal to zero in □□is the golden number □.

9 Conclusions

A series of complex numbers □□, □$(n \square 2)$□□$(n \square 1)$□□(n)□□□□calculated with the formula $\square(n) = \square(n \square 2) \quad \square\square(n \square 1)$, where □ is a complex constant, is studied with the complex variable $CV(n) = \square(n)\square\square(n \square 1)$. The study of $CV(n)$ leads to an Iterated Function System, independent of the calculation of each of the terms of the series, and therefore the algebra of big numbers is avoided. Numerical results show di□erent attractors for di□erent values of □. Probability distributions of the modulus and of the arguments of *CV(n)* are characterized by the existance of peaks and valleys; some of these deep canyons denote very low probability to occur, or none at all. One particular result of interest is that the mean value of the modulus is the golden ratio $\square = \frac{1}{2}\left(1 + \sqrt{5}\right)$ for $\square = e^{i\,\square\square 4}$.

In spite of the non-deterministic nature of the series, a remarkable order is found with the study of the behavior of $CV(n+1)$ as a function of *CV(n)*.

If the terms of the series are real, instead of complex, the self similarity of the probability distribution of □$CV(n)$□is proven with the Stern-Brocot tree. The mean value and the most frequent value of □$CV(n)$□also tend to the golden ratio.

If the sign of the above mentioned series is eliminated by assuming that terms are always added, keeping the complex nature of □ and of the terms of the sequence, a complete landscape of the ratio *CV(n)* in the complex plane is obtained with all possible values of □ . One single point of this panorama, lying along the real axis and corresponding to □ = 1, is the ancient golden ratio □.

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