Differential geometric solution to a classical problem in electrostatics

G. A. Estévez-Bretón

Department of Mathematics and Physical Sciences
Inter American University
00683 San Germán, Puerto Rico

Peilian Lee

Basic Science Department
Anshan Institute of Iron and Steel Technology
Anshan, Lioning
114002 People's Republic of China

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Abstract: *In this article, the normal derivative of the normal component of the macroscopic electrostatic field near the surface of a curved conductor is obtained employing a differential geometric approach.*

Key words: *electrostatic field, normal component, differential geometry*

Resumo: *Empregando-se métodos da geometria diferencial, obtém-se a derivada normal da componente normal do campo eletrostático macroscópico nas proximidades de um condutor de superfície não plana.*

Palavras-chave: *campo eletrostático, componente normal, geometria diferencial*

The general problem of the distribution of charge on the surface of a curved conductor in static electric equilibrium has received some attention during the past two decades [1, 2]. Because the charge density is proportional to the normal component of the electric field, a formula relating the normal derivative of the normal component of the electric field near the conductor to its mean curvature appears to be of interest. There is a classical formula for the rate at which the magnitude of the static electric field decreases with distance away from the surface of a conductor and is given in standard electromagnetism texts [3, 4]

$$\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) E \tag{1}$$

where R_1, R_2 are the two principal radii of curvature of the surface at the point in question. A proof of equation (1) employing a power series expansion has been part of the literature for the past decade [5]. Employing differential geometry we show in this article that the formula for the normal derivative of the normal component of the electric field is given by the expression

$$\frac{\partial E_n}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) E_n \tag{2}$$

We begin our discussion by taking a point P on an equipotential surface *just outside* the conductor as the origin of our coordinate system, and orienting the z axis along the outward normal to the equipotential. Since the electrostatic potential $\Phi(x, y, z)$ on the conductor is a constant, the equipotential passing through the origin may be regarded as the conductor surface itself. Furthermore, the value of Φ for the conductor surface may be taken to be zero without loss of generality. At the origin P we thus have

$$\begin{aligned} E_z(P) &= -\frac{\partial \Phi}{\partial z} = -\Phi_z(P), \\ E_x(P) &= -\Phi_x(P) = 0, \\ E_y(P) &= -\Phi_y(P) = 0 \end{aligned} \tag{3}$$

We now calculate the mean curvature K of the equipotential $\Phi(x, y, z) = 0$ at the origin P. The expression for the mean curvature K is, from differential geometry [6]

$$K = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2} \tag{4}$$

In equation (4) the quantities E, F and G are the coefficients of the first fundamental form whereas L, M and N are the corresponding coefficients of the second fundamental form [6]. Assume now that the surface is described by the parametric equation $r(u, v)$, where r is the radius vector; with the convention that the convex surfaces have a positive curvature, then

$$\begin{array}{llllll} L = & -\mathbf{r}_{uu} \cdot \widehat{\mathbf{n}}, & M = & -\mathbf{r}_{uv} \cdot \widehat{\mathbf{n}} & N = & -\mathbf{r}_{vv} \cdot \widehat{\mathbf{n}}, \\ E = & \mathbf{r}_u \cdot \mathbf{r}_u, & F = & \mathbf{r}_u \cdot \mathbf{r}_v, & G = & \mathbf{r}_v \cdot \mathbf{r}_v \end{array} \tag{5}$$

where $\widehat{\mathbf{n}}$ is the outer unit normal vector:

$$\widehat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{6}$$

With $\Phi(x, y, z) = 0$ and solving for $z = f(x, y)$ we obtain the parametric representation for the surface $\Phi(x, y, z) = 0$:

$$\mathbf{r}\,(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x, y)\,\mathbf{k} \tag{7}$$

where x and y play respectively the roles of the parameters u and v. We can thus write

$$\mathbf{r}_x = \ \mathbf{i} + f_x\,\mathbf{k}, \qquad \mathbf{r}_y \ = \mathbf{j} + f_y\,\mathbf{k} \tag{8}$$

$$\mathbf{r}_{xx} \ = f_{xx}\,\mathbf{k} \qquad \mathbf{r}_{xy} \ = f_{xy}\,\mathbf{k} \qquad \mathbf{r}_{yy} = f_{yy}\,\mathbf{k} \tag{9}$$

As can readily verified, the several partial derivatives of $f(x, y)$ expressed in terms of the derivatives $\Phi(x, y, z)$ are:

$$\begin{aligned} f_x &= -\frac{\partial\Phi/\partial x}{\partial\Phi/\partial z} = -\frac{\Phi_x}{\Phi_z}, \quad f_y = -\frac{\Phi_y}{\Phi_z} \\ f_{xx} &= -\frac{\Phi_{xx}}{\Phi_z} - \frac{\Phi_x^2\Phi_{zz}}{\Phi_z^3} + \frac{2\Phi_x\Phi_{xz}}{\Phi_z^2} \end{aligned} \tag{10}$$

$$\begin{aligned} f_{yy} &= -\frac{\Phi_{yy}}{\Phi_z} - \frac{\Phi_y^2\Phi_{zz}}{\Phi_z^3} + \frac{2\Phi_y\Phi_{zy}}{\Phi_z^2} \\ f_{xy} &= -\frac{\Phi_{xy}}{\Phi_z} - \frac{\Phi_x\Phi_y\Phi_{zz}}{\Phi_z^3} + \frac{\Phi_x\Phi_{yz} + \Phi_y\Phi_{xz}}{\Phi_z^2} \end{aligned} \tag{11}$$

From equations (5)-(11) we obtain for the fundamental forms coefficients *at the point* P, where equation (3) holds true, the following simple expressions:

$$L = \ -f_{xx} = \frac{\Phi_{xx}}{\Phi_z}, \qquad N = -f_{yy} = \frac{\Phi_{yy}}{\Phi_z}, \qquad M = \frac{\Phi_{xy}}{\Phi_z} \tag{12}$$

$$E = G = 1, \qquad F = 0 \tag{13}$$

Equation (4) develops then into

$$\begin{aligned} K &= \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\,(L + N) = -\frac{1}{2}\,(f_{xx} + f_{yy}) \\ K &= \frac{1}{2}\frac{\Phi_{xx} + \Phi_{yy}}{\Phi_z} \end{aligned} \tag{14}$$

In charge-free space the function Φ satisfies Laplace equation $\nabla^2\Phi = 0$. We can thus substitute $\Phi_{xx} + \Phi_{yy}$ in equation (14) by $-\Phi_{zz}$. Then

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{\Phi_{zz}}{\Phi_z} \tag{15}$$

Since

$$E_n = E_z = -\Phi_z, \qquad \frac{\partial E_n}{\partial n} = \frac{\partial E_z}{\partial z} = -\Phi_{zz} \tag{16}$$

then

$$\frac{\partial E_n}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) E_n$$

We now show that equation (1) can be readily deduced. Indeed, since at point P

$$E^2 = E_x^2 + E_y^2 + E_z^2 = E_z^2 \tag{17}$$

then

$$2E\frac{\partial E}{\partial z} = 2E_z\frac{\partial E_z}{\partial z} \tag{18}$$

Thus

$$\frac{\partial E}{\partial n} \equiv \frac{\partial E}{\partial z} = \frac{E_z}{E}\frac{\partial E_z}{\partial z} \tag{19}$$

From equations (2), (16), (17) and (19) we arrive at the result

$$\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)E \tag{20}$$

We thus see that notwithstanding the well-known nonlocal nature of solutions of Laplace equation, and the local dependence of curvature on the shape of the surface, there can be some relationship between the two. The relationship between the partial derivatives of $f(x, y)$ and those of $\Phi(x, y, z)$ are the ones that essentially play the role of the power series expansion of $\Phi(x, y, z)$ in reference [5]. We hope that the differential geometric approach employed in the present article for calculating the normal derivative of the static electric field near the surface of a curved conductor is not only natural and convenient but that it will also be of use to junior students of physics and engineering.

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