Simulation of function $Y = f(X)$ by means of random choices of increments ΔX and ΔY (Method of Expectancies)

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Abstract: *In many mathematical models of growth phenomena, an elemental area ΔA is added to an object growing in the plane; in doing so, the perimeter of the object changes with the area. If ΔA is an elemental area (a square of sides equal to unity), it turns out that the changes of perimeter, ΔP, may have only five possible values: $\Delta P = 4, 2, 0, -2, -4$, depending upon the place where ΔA is added to the cluster. Thus, the function relating the area and the perimeter, $A = f(P)$, may be predicted if the probabilities of the different changes of perimeter are known (or measured). During the aggregation of the n-th particle, the area and the perimeter will be*

$$A_{n+1} = A_n + \Delta A \text{ and } P_{n+1} = P_n + \Delta P$$

respectively. We will herein present the method used with success in growth phenomena but in a more general fashion. We assume that we try to generate any function, $Y = f(X)$, by means of any (finite) number of increments ΔX and ΔY chosen at random from a given set of possibilities for each of them. Thus, the purpose of this paper is the study of the algorithm

$$Y_{n+1} = Y_n + \Delta Y \quad \text{and} \quad X_{n+1} = X_n + \Delta X$$

at the n-th step of the growth of the function.

Key words: *expectancies, growth phenomena*

1. Introduction

Let us assume that there is a broken line in a plane joining the points $P_1(X_1, Y_1)$ and $P_2(X_2, Y_2)$. In going from X_1 to X_2, along the direction of the X-axis, a certain amount n_1 of increment ΔX_1 have been chosen at random; there is also a number n_2 of increments ΔX_2, n_3 of ΔX_3, etc. In general there will be a number n_i of different increments ΔX_i. Thus, the segment $X_2 - X_1$ will be composed of $n_i(\Delta X_i)$ segments; each of the ΔX_i are chosen at random; the increments ΔX_i may have positive or negative values, including zero. The total (finite) number of increments ΔX_i is i_f. If the increments ΔX_i are abundant, and much smaller than $X_2 - X_1$, it may be said that

$$\Delta X = X_2 - X_1 = \sum_{i=1}^{i_f} \Delta X_i n_i(\Delta X_i) \tag{1.1}$$

The same reasoning may be applied to the increments of the ordinates in going from $Y_1$to Y_2 with increments ΔY_j

$$\Delta Y = Y_2 - Y_1 = \sum_{j=1}^{j_f} \Delta Y_j n_j(\Delta Y_j) \tag{1.2}$$

where j_f is the total and finite number of different increments.

The ratio between the two previous equations is

$$\frac{\Delta Y}{\Delta X} = \frac{\sum_{j=1}^{j_f} \Delta Y_j n_j(\Delta Y_j)}{\sum_{i=1}^{i_f} \Delta X_i n_i(\Delta X_i)} \tag{1.3}$$

If n_f is the total amount of steps, *i.e.,* if it is assumed that

$$\sum_{i=1}^{i_f} n_i(\Delta X_i) = \sum_{j=1}^{j_f} n_j(\Delta Y_j) = n_f \tag{1.4}$$

and if the probabilities are defined as

$$p_i(\Delta X_i) = \frac{1}{n_f} \sum_{i=1}^{i_f} n_i(\Delta X_i) \quad \text{and} \quad p_j(\Delta Y_j) = \frac{1}{n_f} \sum_{j=1}^{j_f} n_j(\Delta Y_j) \tag{1.5}$$

then, the following differential equation may be written

$$\frac{\Delta Y}{\Delta X} = \frac{\sum_{j=1}^{j_f} \Delta Y_j p_j(\Delta Y_j)}{\sum_{i=1}^{i_f} \Delta X_i p_i(\Delta X_i)} = \frac{F(X,Y)}{G(X,Y)} \tag{1.6}$$

If the functions $F(X,Y)$ and $G(X,Y)$ are known, the previous equation may be perhaps integrated by any of the known numerical or analytical methods of integration.

In the examples of application below, we will restrict the number of increments to $\Delta X_i = 1, -1$ and 0, and $\Delta Y_j = 1, -1$ and 0; *i.e.*, $i_f = j_f = 3$. Furthermore in order to simplify the presentation of the method, we will only consider functions *G(Y)* and *F(X)*; thus, the two integrals to be solved are

$$\int_{Y_0}^{Y} G(Y)dY = \int_{X_0}^{X} F(X)dX \tag{1.7}$$

or

$$\int_{Y_0}^{Y} \left[p(\Delta X = 1) - p(\Delta X = -1)\right] dY = \int_{X_0}^{X} \left[p(\Delta Y = 1) - p(\Delta Y = -1)\right] dX \tag{1.8}$$

where X_0 and Y_0 are initial conditions. The two terms contained between each of the two brackets in the previous equation are the probabilities of choosing $\Delta X = 1$, $\Delta X = -1$, $\Delta Y = 1$ or $\Delta Y = -1$. In the following, we will give some examples of application of Eq. (1.8), with increasing complexity.

2. The linear function

Let us assume (upper part of figure 2.1) that all the probabilities of the increments ΔX and ΔY are constants

$$p(\Delta X = 1) = C_1 = 0.6,\ p(\Delta X = -1) = C_2 = 0.1$$

and

$$p(\Delta Y = 1) = C_3 = 0.55,\ p(\Delta Y = -1) = C_4 = 0.25$$

with the obvious conditions that

$$p(\Delta X = 0) = 1 - \left[p(\Delta X = 1) + p(\Delta X = -1)\right] = 0.3$$

and

$$p(\Delta Y = 0) = 1 - \left[p(\Delta Y = 1) + p(\Delta Y = -1)\right] = 0.2$$

Upon integration of Eq.(1.8), the linear function becomes

$$Y = \frac{p(\Delta Y = 1) - p(\Delta Y = -1)}{p(\Delta X = 1) - p(\Delta X = -1)} X = \frac{C_3 - C_4}{C_1 - C_2} X = 0.6\, X \tag{2.1}$$

if the initial conditions are

$$X_0 = Y_0 = 0 \tag{2.2}$$

The equation above may be considered as a 'theoretical' expression. We may perform a numerical experiment (shown in the lower part of figure 2.1), choosing the increments ΔX and ΔY at random with the given probabilities. The difference between theory and experiment is quite small; it may be measured with a standard deviation of the form

$$\sigma = \frac{1}{Y_F} \sqrt{\frac{\sum_{n=1}^{N_F} (Y_{theor,n} - Y_{\exp,n})^2}{N_F - 1}} = 0.0052 \tag{2.3}$$

In this particular example, the straight line has performed a total of $n = N_F = 235839$ steps in order to reach $Y = Y_F$ from $Y_0 = 0.$

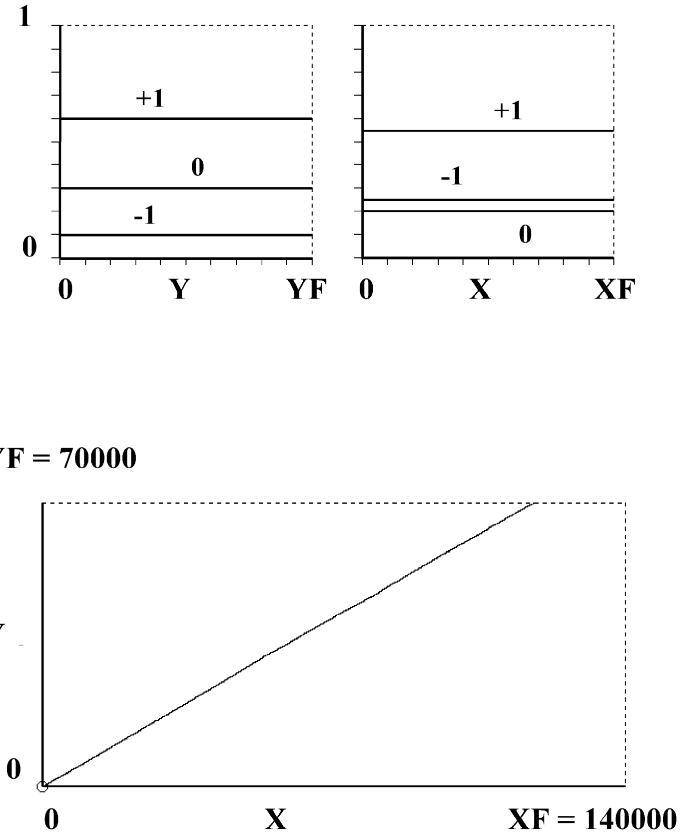

Figure 2.1: The lower sector of the figure is a straight line simulated with the Method of Expectancies, with increments $\Delta X = 1$, $\Delta X = -1$ and $\Delta X = 0$; $\Delta Y = 1$, $\Delta Y = -1$ and $\Delta Y = 0$ chosen at random. The probabilities of increments $p(\Delta X) = f(Y)$ are in the upper left side of the figure, and the probabilities $p(\Delta Y) = f(X)$ in the right side; the values of the increments are indicated with $1, -1$ or 0. The constant set of probabilities are expressed in the text. The theoretical solution is in Eq.(2.1), with the initial condition $X_0 = Y_0 = 0.$ The error between theory and numerical experiment, Eq.(2.3), is small. The total number of steps used to draw this linear function is $N_F = 235839.$

Notice that with a certain amount of steps, N_F, the straight line reaches a point in space. If at this stage of the development the different probabilities of changes of ΔX and ΔY are modified, and if the final point reached by the previous straight line is considered as the starting point of another straight line, then we will obtain a broken line. If the procedure is repeated a specific number of times, any figure in the plane, composed of straight lines, may be represented with this Method of Expectancies. In order to reproduce the given figure, a small amount of information is required since most of the work is done at random.

3. A quadratic equation

Let us think now of an example in which the probabilities of increments ΔX are linear functions of Y, and the probabilities of increments ΔY are constant. The assigned values for ΔX are

$$\begin{aligned} p(\Delta X = 1) &= p_{in}(\Delta X = 1) + [p_{final}(\Delta X = 1) - p_{in}(\Delta X = 1)]\,(Y/Y_F) \\ &= 0.6 - 0.6(Y/70000) \end{aligned} \tag{3.1}$$

$$\begin{aligned} p(\Delta X = -1) &= p_{in}(\Delta X = -1) + [p_{final}(\Delta X = -1) - p_{in}(\Delta X = -1)]\,(Y/Y_F) \\ &= 0.9(Y/70000) \end{aligned} \tag{3.2}$$

$$\begin{aligned} p(\Delta X = 0) &= p_{in}(\Delta X = 0) + [p_{final}(\Delta X = 0) - p_{in}(\Delta X = 0)]\,(Y/Y_F) \\ &= 0.4 - 0.3(Y/70000) \end{aligned} \tag{3.3}$$

and those for ΔY are

$$p(\Delta Y = 1) = C_1 = 0.4 \quad \text{and} \quad p(\Delta Y = -1) = C_2 = 0.35 \tag{3.4}$$

with

$$p(\Delta Y = 0) = 1 - [p(\Delta Y = 1) + p(\Delta Y = -1)] = 0.25$$

Solving the two integrals of Eq.(1.8), with the probabilities given in Eqs.(3.1) through (3.4), the following quadratic equation is obtained

$$\begin{aligned} &\int_{Y_0}^{Y} [p(\Delta X = 1) - p(\Delta X = -1)]dY = [p_{in}(\Delta X = 1) = p_{in}(\Delta X = -1)] \int_{Y_0}^{Y} dY + \\ &\frac{1}{Y_F}[p_{final}(\Delta X = 1) - p_{in}(\Delta X = 1) + p_{in}(\Delta X = -1) - p_{final}(\Delta X = -1)] \int_{Y_0}^{Y} Y dY \\ &= 0.6(Y - Y_0) - 1.0714 \times 10^{-5}(Y^2 - Y_0^2) \end{aligned} \tag{3.5}$$

for the integral at the left hand side of Eq.(1.8). For the integral at the right hand side we obtain

$$\int_{X_0}^{X} [p(\Delta Y = 1) - p(\Delta Y = -1)]dX = \\ (C_1 - C_2)(X - X_0) = 0.05(X - X_0) \tag{3.6}$$

The resulting equation is

$$0.6(Y - Y_0) - 1.0714 \times 10^{-5}(Y^2 - Y_0^2) = 0.05(X - X_0) \tag{3.7}$$

If Eq.(3.7) is regarded as the theoretical view of the Method (shown with circles in the lower part of figure 3.1), the numerical experiment (full line in the same figure) fits the theory with reasonable accuracy. One way to measure the error is by means of the definition

$$\varepsilon = \frac{1}{X_F}\sqrt{\frac{\sum_{n=1}^{N_F}(X_{theor,n} - X_{\exp,n})^2}{N_F - 1}} = 0.0065 \tag{3.8}$$

The sector of the curve where there is a minimum radius of curvature (near the maximum reach of the curve along the X-axis), corresponds to the 'time' when probabilities to choose $\Delta X = 1, -1$ or 0 are nearly the same. Even in this unfavorable sector of the curve, the method seems to work with good accuracy.

4. A set of hyperbolas

We will work out now an example in which both ΔX and ΔY are linear functions of Y for the former and of X for the later. The functions are

$$p(\Delta X = 1) = p_{in}(\Delta X = 1) + [p_{final}(\Delta X = 1) - p_{in}(\Delta X = 1)]\frac{Y}{Y_F} \\ = 0.7 + (0.3 - 0.7)\frac{Y}{70000} = 0.7 - 0.4\frac{Y}{70000} \tag{4.1}$$

$$p(\Delta X = -1) = p_{in}(\Delta X = -1) + [p_{final}(\Delta X = -1) - p_{in}(\Delta X = -1)]\frac{Y}{Y_F} \\ = 0 + (0.5 - 0)\frac{Y}{70000} = 0.5\frac{Y}{70000} \tag{4.2}$$

$$p(\Delta Y = 1) = p_{in}(\Delta Y = 1) + [p_{final}(\Delta Y = 1) - p_{in}(\Delta Y = 1)]\frac{X}{X_F} \\ = 0.6 + (0.3 - 0.6)\frac{X}{140000} = 0.6 - 0.3\frac{X}{140000} \tag{4.3}$$

$$p(\Delta Y = -1) = p_{in}(\Delta Y = -1) + [p_{final}(\Delta Y = -1) - p_{in}(\Delta Y = -1)]\frac{X}{X_F} \\ = 0.2 + (0.6 - 0.2)\frac{X}{140000} = 0.2 + 0.4\frac{X}{140000} \tag{4.4}$$

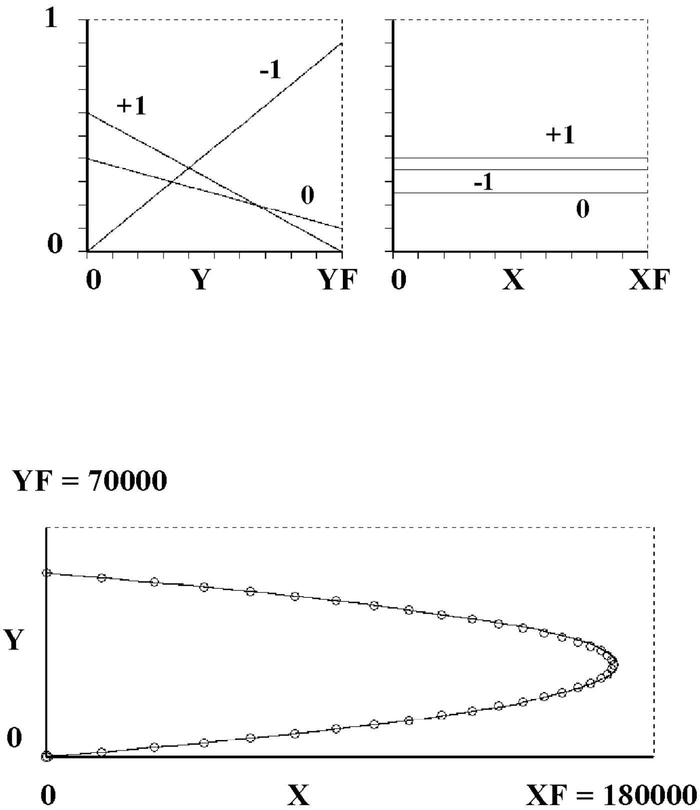

Figure 3.1: A quadratic function, as computed with the Method of Expectancies, is shown in the lower part of the figure. It results from increments chosen at random with probabilities $p(\Delta X) = f(Y)$ (upper left) and $p(\Delta Y) = f(X)$ (upper right). The different values of the increments are indicated with +1 ,-1 and 0. Probabilities are expressed by Eqs.(3.1) through (3.4). The theoretical solution, Eq.(3.7), is shown with open circles. The small error of Eq.(3.8) denotes that the fit is quite satisfactory.

One of the integrals of Eq.(1.8) is

$$\int_{Y_0}^{Y} [p(\Delta X = 1) - p(\Delta X = -1)]\, dY = \int_{Y_0}^{Y} [p_{in}(\Delta X = 1) - p_{in}(\Delta X = -1)]\, dY +$$

$$\frac{1}{Y_F}\int_{Y_0}^{Y} \{[p_{final}(\Delta X = 1) - p_{in}(\Delta X = 1)] - [p_{final}(\Delta X = -1) - p_{in}(\Delta X = -1)]\} Y\, dY =$$

$$0.7(Y - Y_0) + \frac{-0.9}{2 \times 70000}(Y^2 - Y_0^2) = 0.7(Y - Y_0) - 6.4286 \times 10^{-6}(Y^2 - Y_0^2) \qquad (4.5)$$

The other integral is

$$\int_{X_0}^{X} [p(\Delta Y = 1) - p(\Delta Y = -1)]\, dX = \int_{X_0}^{X} [p_{in}(\Delta Y = 1) - p_{in}(\Delta Y = -1)]\, dX +$$
$$\frac{1}{X_F}\int_{X_0}^{X} \{[p_{final}(\Delta Y{=}1) - p_{in}(\Delta Y{=}1)] - [p_{final}(\Delta Y{=}-1) - p_{in}(\Delta Y{=}-1)]\} X\, dX =$$
$$(0.6 - 0.2)(X - X_0) + \frac{-0.7}{2 \cdot 140000}(X^2 - X_0^2) = 0.4(X - X_0) - 2.5 \cdot 10^{-6}(X^2 - X_0^2) \tag{4.6}$$

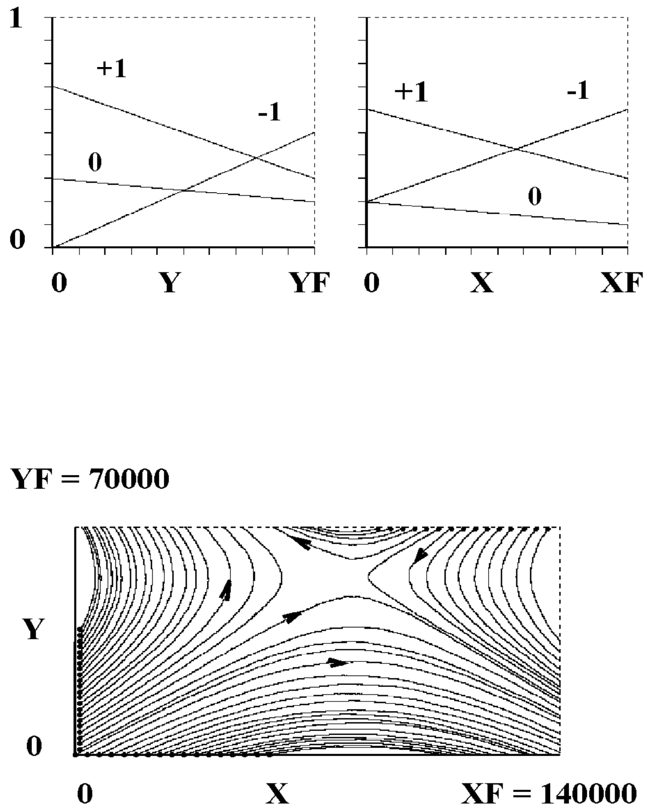

Figure 4.1: A numerical model chooses increments $\Delta X = 1, -1$ or 0 at random with probabilities $p(\Delta X) = f(Y)$: they are varying linearly with Y, as shown in the upper left sector. In a similar fashion, increments $\Delta Y = 1, -1$or 0 are also chosen at random with probabilities $p(\Delta Y) = f(X)$: they are also varying linearly but with X, shown in the upper right sector; probabilities are in Eqs.(4.1) through (4.4). The result is the set of hyperbolas in the lower part of the figure. The use of the Method of Expectancies yields Eqs.(4.5) and (4.6), quite well approximated by numerical results.

The theoretical Eqs.(4.5) and (4.6) are represented with a numerical experiment in the lower part of figure 4.1. The upper part of this figure shows the probabilities by means of which the six different increments are randomly chosen. Hyperbolas start from different points $P_0(X_0, Y_0)$ indicated with full circles; the direction of growth of the curves is represented with small arrows. These curves have two straight lines (tangent to the hyperbolas) which intersect each other at one point. A numerical experiment can be performed with the Method of Expectancies in order to draw the two tangents, provided their two starting points (X_0, Y_0) are suitably selected by means of ordinary analytical procedures. If one experiment starts from the lower left sector (in figure 4.1) the zigzagging curve will follow the tangent with reasonable accuracy when it is rather far from the point of intersection; the same will occur when another numerical experiment starts near the upper right corner of figure 4.1. Nevertheless, when both experiments are very near the point of intersection they will deviate from the tangents. In some cases the deviation (by chance) is small and in other they miss the point of intersection with curved paths. These random instabilities are due to the fact that the probabilities of choosing $\Delta X = 1$ or -1 and $\Delta Y = 1$ or -1 are nearly the same in the neighborhood of the intersection of tangents. It is very likely that the random process near the intersection of tangents exhibit some properties of chaos.

5. A trigonometric function

For the left part of Eq.(1.8) we select the constant values

$$p(\Delta X = 1) = C_1 = 0.6, \qquad \text{and} \qquad p(\Delta X = -1) = C_2 = 0.3 \tag{5.1}$$

and the result of the integration becomes

$$\int_{Y_0}^{Y} [p(\Delta X = 1) - p(\Delta X = -1)]\, dY = (C_1 - C_2)(Y - Y_0) \tag{5.2}$$

For the increments ΔY we will choose

$$\begin{aligned} p(\Delta Y &= 1) = p_{mean}(\Delta Y = \pm 1) + p_{ampl}(\Delta Y = 1) \sin\left(\frac{2\pi X}{X_F}\right) \\ &= 0.4 + 0.2 \sin\left(\frac{2\pi X}{X_F}\right) \end{aligned} \tag{5.3}$$

$$\begin{aligned} p(\Delta Y &= -1) = p_{mean}(\Delta Y = \pm 1) + p_{ampl}(\Delta Y = -1) \sin\left(\frac{2\pi X}{X_F}\right) \\ &= 0.4 - 0.2 \sin\left(\frac{2\pi X}{X_F}\right) \end{aligned} \tag{5.4}$$

both with $X_F = 300000$. p_{mean} is a mean probability in which it is mounted a sinusoidal function of maximum amplitude given by p_{ampl}. Then the integral in the right hand side of Eq.(1.8) becomes

$$\begin{aligned}
&\int_{X_0}^{X} [p(\Delta Y = 1) - p(\Delta Y = -1)]\, dX = [p_{mean}(\Delta Y = 1) - p_{mean}(\Delta Y = -1)] \int_{X_0}^{X} dX \\
&+[p_{ampl}(\Delta Y = 1) - p_{ampl}(\Delta Y = -1)] \int_{X_0}^{X} \sin\left(\frac{2\pi X}{X_F}\right) dX \\
&= (0.4 - 0.4)(X - X_0) + \left(\frac{X_F}{2\pi}\right)(0.2 + 0.2)\left[\cos\left(\frac{2\pi X_0}{X_F}\right) - \cos\left(\frac{2\pi X}{X_F}\right)\right] \\
&= 1.9099 \times 10^4 \left[1 - \cos\left(\frac{2\pi X}{X_F}\right)\right]
\end{aligned} \tag{5.5}$$

The resulting theoretical solution

$$Y_{theor} = 6.3662 \times 10^4 \left[1 - \cos\left(\frac{2\pi X}{X_F}\right)\right] \tag{5.6}$$

is shown with small circles in the lower part of figure 5.1. If the increments ΔX and ΔY are chosen at random, with the assigned probabilities, shown in the upper part of figure 5.1 and given in Eqs.(5.1), (5.3) and (5.4), a numerical experiment may be performed. It may be clearly seen that the experiment (full line) fits quite well the theoretical results given by Eq.(5.6).

An error may be defined as

$$\varepsilon = \frac{1}{Y_{\max}} \sqrt{\frac{\sum_{n=1}^{N_F} (Y_{theor,n} - Y_{\exp,n})^2}{N_F - 1}} = 0.00354 \tag{5.7}$$

The reference of the error is $Y_{\max}$, equal to twice the amplitude of the function; we have covered two cycles of the cosine function, with a total of $N_F = 2 \times 10^6$ steps.

6. A set of self-avoiding curves

For the next example, we will choose the following sinusoidal variation of probabilities

$$\begin{aligned}
p(\Delta X &= 1) = p_{mean}(\Delta X = 1) + p_{ampl}(\Delta X = 1) \sin\left(\frac{2\pi Y}{Y_F}\right) \\
&= 0.3 + 0.25 \sin\left(\frac{2\pi Y}{Y_F}\right)
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
p(\Delta X &= -1) = p_{mean}(\Delta X = -1) + p_{ampl}(\Delta X = -1) \sin\left(\frac{2\pi Y}{Y_F}\right) \\
&= 0.4 - 0.3 \sin\left(\frac{2\pi Y}{Y_F}\right)
\end{aligned} \tag{6.2}$$

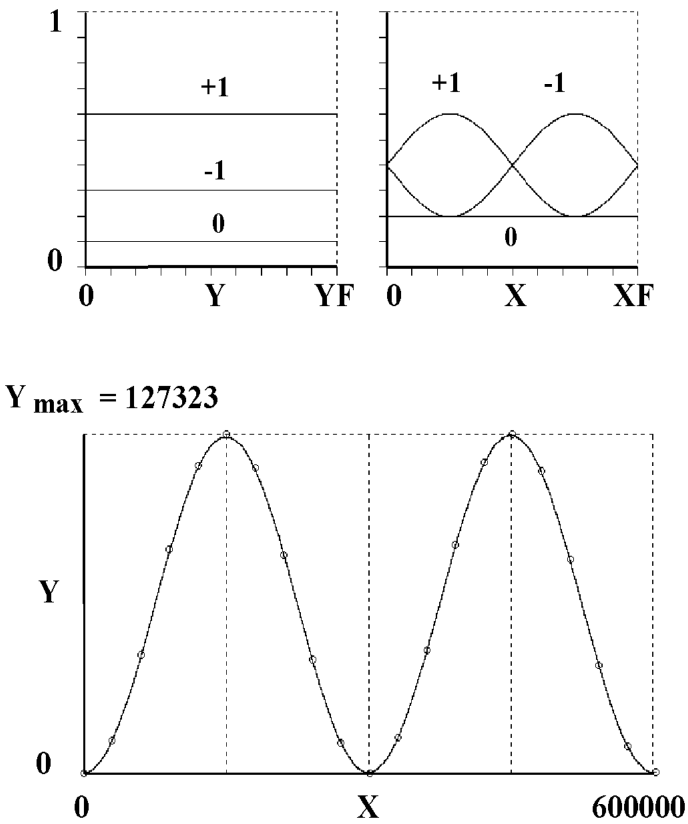

Figure 5.1: A trigonometric function, Eq.(5.6) with open circules, is reasonably represented with the Method of Expectancies if probabilities of increments are given by Eqs.(5.1), (5.3) and (5.4), in the upper part of the figure. The numerical integration demands $N_F = 2 \times 10^6$ steps, with the small error of Eq.(5.7).

With these two expressions the left hand side of Eq.(1.8) becomes

$$\int_{Y_0}^{Y} [p(\Delta X{=}1) - p(\Delta X{=}-1)]\, dY = [p_{mean}(\Delta X{=}1) - p_{mean}(\Delta X{=}-1)] \int_{Y_0}^{Y} dY$$
$$+ [p_{ampl}(\Delta X = 1) - p_{ampl}(\Delta X = -1)] \int_{Y_0}^{Y} \sin\left(\frac{2\pi Y}{Y_F}\right) dY$$
$$= (0.3 - 0.4)(Y - Y_0) + \left(\frac{Y_F}{2\pi}\right)(0.25 + 0.3)\left[\cos\left(\frac{2\pi Y_0}{Y_F}\right) - \cos\left(\frac{2\pi Y}{Y_F}\right)\right]$$
$$= -0.1(Y - Y_0) + 4.3768 \times 10^3 \left[\cos\left(\frac{2\pi Y_0}{Y_F}\right) - \cos\left(\frac{2\pi Y}{Y_F}\right)\right] \tag{6.3}$$

With respect to the right hand side of Eq.(1. 8), we define

$$
\begin{aligned}
p(\Delta Y &= 1) = p_{mean}(\Delta Y = 1) + p_{ampl}(\Delta Y = 1)\sin\left(\frac{2\pi X}{X_F}\right)\\
&= 0.5 + 0.2\sin\left(\frac{2\pi X}{X_F}\right)
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
p(\Delta Y &= -1) = p_{mean}(\Delta Y = -1) + p_{ampl}(\Delta Y = -1)\sin\left(\frac{2\pi X}{X_F}\right)\\
&= 0.3 - 0.2\sin\left(\frac{2\pi X}{X_F}\right)
\end{aligned}
\tag{6.5}
$$

by means of which

$$
\begin{aligned}
&\int_{X_0}^{X}[p(\Delta Y{=}1) - p(\Delta Y{=}-1)]\,dX = [p_{mean}(\Delta Y{=}1) - p_{mean}(\Delta Y = -1)]\int_{X_0}^{X} dX\\
&\qquad + [p_{ampl}(\Delta Y = 1) - p_{ampl}(\Delta Y = -1)]\int_{X_0}^{X}\sin\left(\frac{2\pi X}{X_F}\right)dX\\
&= (0.5 - 0.3)(X - X_0) + \left(\frac{X_F}{2\pi}\right)(0.2 + 0.2)\left[\cos\left(\frac{2\pi X_0}{X_F}\right) - \cos\left(\frac{2\pi X}{X_F}\right)\right]\\
&= 0.2(X - X_0) + 3.1831\times 10^{3}\left[\cos\left(\frac{2\pi X_0}{X_F}\right) - \cos\left(\frac{2\pi X}{X_F}\right)\right]
\end{aligned}
\tag{6.6}
$$

With the probabilities given in Eqs.(6.1), (6.2), (6.4) and (6.5), shown in the up part of figure 6.1, we have performed numerical experiments with the Method of Expectancies in a field of $X_F = Y_F = 50000$. The results of 100 curves are shown in the lower part of figure 6.1; each of them starts along $Y_0 = -Y_F$ and in the range $-2X_F \le X_0 \le 2X_F$. The distance between each origin of the curves is $X_F/25$; each of the curves is allowed to perform $N_F = 10^6$ steps of integration. The horizontal field of the lower part of figure 6.1 is $4X_F = 200000$ and the vertical size is $2.5Y_F = 125000$, approximately.

The other type of curves are closed loops. Four of them ('eyes') are clearly visible in the lower part of figure 6.1, and they are well defined because many curves started at $Y_0 = -Y_F$; the rest of the eyes are empty because no closed loops started in the inside of the eyes. In the vicinity of the closed loops there are regions of high instability due to the fact that probabilities of increments of one class are approximately equal (or equal) to the probabilities of increments of another class.

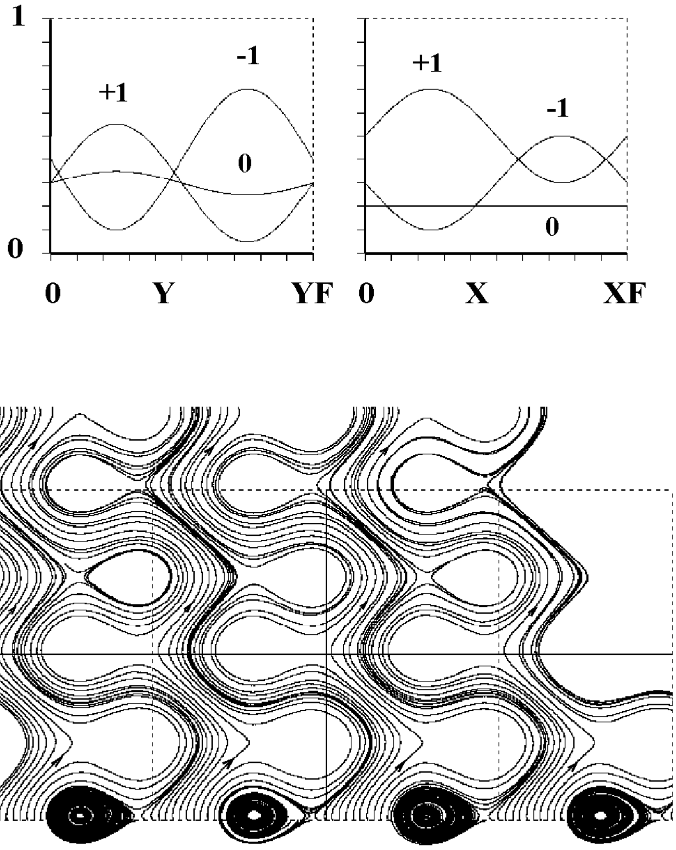

Figure 6.1 The set of six periodic probabilities of increments given by Eqs.(6.1), (6.2), (6.4) and (6.5) (in the upper part of the figure), yield a set of self-avoiding curves when the method herein proposed is used. The theoretical solution is in Eqs.(6.3) and (6.6). See text for the initial conditions of each curve. It may be clearly seen that there are two types of curves. On the one hand there are self-avoiding 'open' curves; some examples are the curves starting at $Y_0 = -Y_F$ and at $X_0 = -2X_F, -X_F, 0, X_F$ or $2X_F$. These open curves ascend in a wavy fashion towards large and positive values of Y.

These regions of high instability nearby the eyes are due to the fact that the Method of Expectancies, for this particular example, struggles to decide between $\Delta X = 1, 0$ or -1, and between $\Delta Y = 1, 0$ or -1. The points of highest instability, which may be called 'points of indecision', may be found from Eqs.(6.4) and (6.5) by equating $p(\Delta Y = 1) = p(\Delta Y = -1)$ for the horizontal axis. This yields $X_{ind}/X_F = 0.5833$ and $X_{ind}/X_F = 0.9167$. The other two points of indecision along the horizontal axis come from $p(\Delta X = 0) = p(\Delta X = -1)$ with $X_{ind}/X_F = 0.0833$ and $X_{ind}/X_F = 0.4167$.

Points of indecision along the vertical axis can be found with a similar procedure from Eqs.(6.1) and (6.2): $p(\Delta X = 1) = p(\Delta X = -1)$. They are placed at $Y_{ind}/Y_F = 0.0291$ and $Y_{ind}/Y_F = 0.4709$. From the condition $p(\Delta X = 1) = p(\Delta X = 0)$, it is found that $Y_{ind}/Y_F = 0$ and $Y_{ind}/Y_F = 0.5$; and from $p(\Delta X = -1) = p(\Delta X = 0)$, the last pair of points of indecision is $Y_{ind}/Y_F = 0.0461$ and $Y_{ind}/Y_F = 0.4539$.

7. Conclusion

It is proved that a wide variety of functions $Y = f(X)$ can be approximated with the Method of Expectancies, in which the function is built up step by step with increments ΔX and ΔY given in a probabilistic way. The zigzagging line thus obtained (at random) is compared with analytic functions obtained through the integration of the resulting differential equation. The error between theoretical and random numerical experiments is reasonably small. It should also be noticed that very complex patterns of curves may be obtained with scarce initial information.

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